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Operations and Cooperations in Elliptic Cohomology, Part I: Generalized modular forms and the cooperation algebra

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Abstract. This is the first of two interconnected parts: Part I contains the geometric theory of *generalized modular forms* and their connections with the cooperation algebra for elliptic cohomology, $E\ell\ell_*E\ell\ell$, while Part II is devoted to the more algebraic theory associated with Hecke algebras and stable operations in elliptic cohomology.

We investigate the structure of the stable operation algebra $E\ell\ell^*E\ell\ell$ by first determining the dual cooperation algebra $E\ell\ell_*E\ell\ell$. A major ingredient is our identification of the cooperation algebra $E\ell\ell_*E\ell\ell$ with a ring of generalized modular forms whoses exact determination involves understanding certain integrality conditions; this is closely related to a calculation by N. Katz of the ring of all 'divided congruences' amongst modular forms. We relate our present work to previous constructions of Hecke operators in elliptic cohomology. We also show that a well known operator on modular forms used by Ramanujan, Swinnerton-Dyer, Serre and Katz cannot extend to a stable operation.

Introduction

This paper is in two interelated parts: Part I contains the geometric theory of *generalized modular forms* and their connections with the cooperation algebra $E\ell\ell_*E\ell\ell$, while Part II will be devoted to the more algebraic theory associated with Hecke algebras and operations in elliptic cohomology.

In our earlier paper [6], we defined operations in the 'level 1' version of elliptic cohomology $E\ell\ell^*(\ )$ which restricted to the classical Hecke operators on the coefficient ring $E\ell\ell_*$ (defined to be a ring of modular forms for the full modular group $\mathrm{SL}_2(\mathbb{Z})$). In the present paper we investigate the structure of the operation algebra $E\ell\ell^*E\ell\ell$ by determining the dual cooperation algebra $E\ell\ell_*E\ell\ell$, thus following the pattern established in the case of K-theory; we also describe a category

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of modules (dually comodules) over these which are closely related to modules over Hecke algebras associated to the group $\mathrm{SL}_2(\mathbb{Z})$; this points to a generalization from K-theory to elliptic cohomology of work by A. K. Bousfield in [12], [13]. A recent paper of F. Clarke and K. Johnson [14] has also considered the analogous cooperation algebra for the level 2 version of elliptic cohomology, and we in effect prove their conjecture on the structure of their analogue of $E\ell\ell_*E\ell\ell$.

A particular ingredient is our identification of the cooperation algebra $E\ell\ell_*E\ell\ell$ with a ring of 'generalized modular forms'. The most significant aspect of this involves understanding certain integrality conditions, and this is closely related to the calculation by N. Katz in [23] of the ring of all 'divided congruences' amongst modular forms (in 1 variable). Indeed, Katz's work amounts to a calculation of the topological gadget $KU_*E\ell\ell$ rather than $E\ell\ell_*E\ell\ell$; however, we use his results to determine the latter. We also wish to point out that the construction by G. Nishida [32] of Hecke operators appears to be closely related to the ideas of the present work.

We will assume the reader is familiar with the apparatus of algebraic topology contained in [1] and [33], to which the reader is referred for all basic ideas on complex oriented cohomology theories and their associated formal group laws. As basic references on elliptic cohomology theories, P. S. Landweber's two articles [28] and [29] are highly recommended although their main emphasis is on level 2 theories. A more recent reference is that of J. Francke [15]. A convenient source for all the basic notions of Hecke algebras is [26].

In detail, Part I is structured as follows. §1 contains a brief resumé of modular forms and elliptic cohomology. §2 gives details of the formal group law associated to elliptic curves in Weierstrass form and the canonical complex orientation of elliptic cohomology. §3 introduces the cooperation Hopf algebroid $E\ell\ell_*E\ell\ell$. §4 introduces our notion of generalized modular form. In §5 and §6 we describe certain categories of isogenies and their realisation as stable operations on elliptic cohomology. §7 recalls the properties of the classical rings of stably numerical polynomials, familiar in the context of the stable cooperation Hopf algebroid for K-theory, KU_*KU. In §8 and §9 we describe a major result of N. Katz and apply it to the calculation of our ring of generalized modular forms which is isomorphic to $E\ell\ell_*E\ell\ell$. In §10 and §11 we complete the description of $E\ell\ell_*E\ell\ell$ by considering its coproduct structure and use duality to construct stable operations, particularly operations which generalize the classical Hecke operators. Finally, in §12 we discuss an important operation ∂ on modular forms which is a derivation and plays a major rôle in the arithmetic theory of Swinnerton-Dyer, Serre and Katz; we show this cannot extend to a stable operation in elliptic cohomology.

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1. Modular forms and elliptic cohomology

Let $\mathcal{L}$ denote the set of all *oriented lattices in* $\mathbb{C}$, i.e., discrete free subgroups $L \subseteq \mathbb{C}$ such $\mathbb{R} \otimes L = \mathbb{C}$ as oriented real vector spaces. This set can be identified with the coset space

$$\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{V},$$

where $\mathcal{V}$ is the set of all *oriented bases* $\{\omega_1, \omega_2\}$ in the real vector space $\mathbb{C}$ and we use the convention that for an oriented (ordered) basis $\{\omega_1, \omega_2\}$,

$$\omega_1/\omega_2 \in \mathfrak{H} = \{\tau \in \mathbb{C} : \operatorname{im} \tau > 0\}.$$

The action of $\mathrm{SL}_2(\mathbb{Z})$ is the obvious one,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \{\omega_1, \omega_2\} = \{a\omega_1 + b\omega_2, c\omega_1 + d\omega_2\}.$$

This of course induces the usual action on the upper half plane $\mathfrak{H}$ on passage from $\{\omega_1, \omega_2\}$ to ω_1/ω_2. Thus $\mathcal{L}$ possesses a natural 2-dimensional complex analytic structure.

Notice that the group of non-zero complex numbers $\mathbb{C}^\times$ acts compatibly on both $\mathcal{V}$ and $\mathcal{L}$ by

$$\lambda \cdot \{\omega_1, \omega_2\} = \{\lambda\omega_1, \lambda\omega_2\}$$

and

$$\lambda \cdot \langle\omega_1, \omega_2\rangle = \langle\lambda\omega_1, \lambda\omega_2\rangle\,,$$

where $\langle\omega_1, \omega_2\rangle$ denotes the lattice spanned by the basis $\{\omega_1, \omega_2\}$.

We will follow [22] and [25] in defining *a modular form of weight* k to be a holomorphic function $F\colon \mathcal{L} \longrightarrow \mathbb{C}$ which satisfies the functional equation

$$F(\lambda \cdot L) = \lambda^{-k} F(L)$$

whenever $\lambda \in \mathbb{C}^\times$. To avoid excessively elaborate notation, we will sometimes regard such a function as having as its domain $\mathcal{V}$ and being invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$. We can associate to such an F a function $f\colon \mathfrak{H} \longrightarrow \mathbb{C}$ defined by $f(\tau) = F(\langle\tau, 1\rangle)$ for $\tau \in \mathfrak{H}$ the upper half plane. After setting $q = e^{2\pi i\tau}$, we say that F is *holomorphic at infinity* (i.e., *at* $i\infty$) if the Fourier series expansion

$$f(\tau) = \sum_{-\infty<n<\infty} a_n q^n$$

has $a_n = 0$ for $n < 0$; if also $a_0 = 0$, then F is a *cusp form.* We say that F is *meromorphic at infinity* if the Fourier series of F has $a_n = 0$ for $n \ll 0$. If the coefficients a_n lie in some subring $K \subseteq \mathbb{C}$, then we say that F is *defined over* K. Throughout this paper we will assume as we did in [6] that $\mathbb{Z}[1/6] \subseteq K$, the reader is referred to [22] and [42] for details on the reasons for this. We will denote by $\mathrm{S}(K)_k$ the set of all weight k modular forms holomorphic at infinity and by $\mathrm{M}(K)_k$ the set of all weight k modular forms meromorphic at infinity; of course we have $\mathrm{S}(K)_k \subseteq \mathrm{M}(K)_k$. Thus there are two strictly commutative graded rings $\mathrm{S}(K)_*$ and $\mathrm{M}(K)_*$ with a homomorphism of graded rings $\mathrm{S}(K)_* \longrightarrow \mathrm{M}(K)_*$. The following classical result describes the structure of such rings. Elementary accounts of this result can be found in [25, 39]; for a discussion of rigidity under base change, see [22].

Theorem 1.1. *If $1/6 \in K$, then as graded rings we have*

$$\mathrm{S}(K)_* = K[E_4, E_6],$$

and

$$\mathrm{M}(K)_* = \mathrm{S}(K)_*[\Delta^{-1}] = K[E_4, E_6, \Delta^{-1}],$$

where $E_{2n} \in \mathrm{S}(K)_{2n} \subseteq \mathrm{M}(K)_{2n}$ is the $2n$th Eisenstein function *and*

$$\Delta = \frac{1}{1728}\left(E_4^3 - E_6^2\right)$$

is the discriminant function.

We recall the following q-expansions defined over $\mathbb{Q}$:

$$E_{2n}(q) = 1 - \frac{4n}{B_{2n}} \sum_{k \geqslant 1} \sigma_{2n-1}(k) q^k \quad \text{for } n \geqslant 1 \tag{1.1}$$

$$\Delta = q \prod_{n \geqslant 1} (1 - q^n)^{24} \tag{1.2}$$

where $\sigma_m(k) = \sum_{d|k} d^m$. Whenever $n > 1$, the q-expansion $E_{2n}(q)$ corresponds to a modular form of weight $2n$, which we will denote by E_{2n}. Notice that for any subring $K \subseteq \mathbb{C}$, we have $E_4, E_6 \in \mathrm{S}(K)_* \subseteq \mathrm{M}(K)_*$. Following [40, 41], we will use the notation $Q = E_4$ and $R = E_6$.

For each $n \geqslant 0$, define a basis $\{F_{n,a}\}$ of $\mathrm{S}(K)_n$ over K as follows. For $0 \leqslant n \leqslant 14$, set

$$\begin{aligned}
F_{0,0} &= 1, \\
F_{4,0} &= Q = E_4, \\
F_{6,0} &= R = E_6, \\
F_{8,0} &= Q^2, \\
F_{10,0} &= QR, \\
F_{12,0} &= Q^3, \\
F_{12,1} &= \Delta, \\
F_{14,0} &= Q^2 R.
\end{aligned}$$

For $n \geqslant 16$, inductively define the basis so that $F_{n,0} = Q^3 F_{n-12,0}$, and if $a \geqslant 1$, $F_{n,a} = \Delta F_{n-12,a-1}$. Notice that we have

$$F_{m,a} F_{n,b} = \begin{cases} F_{m+n,0} + (\text{cusp form}) & \text{if } a = b = 0, \\ (\text{cusp form}) & \text{otherwise.} \end{cases} \tag{1.3}$$

We will refer to the basis $\{F_{n,a}\}$ as the *standard basis* of the graded K-module $\mathrm{S}(K)_*$. We can lexicographically order this basis by the index (n, a).

We next introduce the following topologically motivated notation:

$$\begin{aligned}
e\ell\ell_{2n} &= \mathrm{S}(\mathbb{Z}[1/6])_n, \\
E\ell\ell_{2n} &= \mathrm{M}(\mathbb{Z}[1/6])_n.
\end{aligned}$$

We define elliptic cohomology to be the functor (on the category of finite CW complexes or spectra)

$$E\ell\ell^*(\ ) = E\ell\ell_* \underset{MU_*}{\otimes} MU^*(\ ). \tag{1.4}$$

In Landweber's papers [28, 29] and also [6], it is shown that this a cohomology theory. There is also a connective theory $e\ell\ell^*(\ )$ whose coefficient ring is $e\ell\ell_*$, although we make no use of it in this paper. However, its representing spectrum $e\ell\ell$ is *not* the connective covering of $E\ell\ell$, even if the notation may suggest this.

We end this section with some further remarks on elliptic cohomology, intended to highlight its properties as a cohomology theory. In [8] we observed that after a suitable completion, the spectrum $E\ell\ell$ carries a unique topological A_∞ ring structure (in unpublished work we have also shown that this is true for $E\ell\ell$ itself). An important consequence of this is that for any A_∞ module spectrum M over $E\ell\ell$ and any spectrum X, there are Künneth and Universal Coefficient spectral sequences for $M_*(X)$ and $M^*(X)$, This depends upon work of C. A. Robinson [35, 36, 37]. An alternative approach to such spectral sequences comes from recent work of M. J. Hopkins and J. R. Hunton [20, 21], whose methods yield the following theorem.

Theorem 1.2. *For any $d \in \mathbb{Z}$, let $\Omega^{\infty-d}E\ell\ell$ denote the term in the Ω-spectrum $E\ell\ell$ which represents the elliptic cohomology group $E\ell\ell^d(\ )$. Then the ordinary homology $H_*(\Omega^{\infty-d}E\ell\ell;\mathbb{Z}[1/6])$ is torsion free. Similarly, $E\ell\ell_*(\Omega^{\infty-d}E\ell\ell)$ is free over $E\ell\ell_*$. Consequently, the spectrum $E\ell\ell$ is a colimit of finite CW spectra E_α each having the property that both $E\ell\ell_*(E_\alpha)$ and $E\ell\ell_*(DE_\alpha)$ are free over $E\ell\ell_*$.*

Recall the conditions for Adams' universal coefficient spectral sequence of [1], Part III.

Corollary 1.3. *The conditions for Adams' universal coefficient spectral sequence are satisfied by the spectrum $E\ell\ell$. Hence the Künneth and Universal Coefficient spectral sequences exist for any module spectrum over $E\ell\ell$ and any spectrum X, and have the usual forms:*

$$\begin{cases} \mathrm{E}^2_{*,*}(X) & \Longrightarrow M_*(X) \\ \mathrm{E}^2_{*,*}(X) & = \mathrm{Tor}^{*,*}_{E\ell\ell_*}(E\ell\ell_*(X), M_*) \end{cases}$$

and

$$\begin{cases} \mathrm{E}_2^{*,*}(X) & \Longrightarrow M^*(X) \\ \mathrm{E}_2^{*,*}(X) & = \mathrm{Ext}^{*,*}_{E\ell\ell_*}(E\ell\ell_*(X), M_*) \end{cases}$$

Thus, elliptic homology and cohomology possess the usual battery of computational technology. However, the fact that the coefficient ring $E\ell\ell_*$ is not a principal ideal domain suggests that serious calculations will usually be of greater difficulty than they would in say K-theory. For reductions modulo invariant ideals and relations with Morava $K(1)$ and $K(2)$, see [8, 9, 10].

We end this section by describing a modified version of elliptic cohomology which is 2-periodic. We take as its coefficient ring

$$\mathcal{E}\ell\ell_* = E\ell\ell_*[\Lambda]/(\Lambda^{12} - \Delta),$$

where $\Lambda \in \mathcal{E}\ell\ell_2$. Then the natural homomorphism $E\ell\ell_* \longrightarrow \mathcal{E}\ell\ell_*$ allows us to define the functors (on finite CW complexes or spectra)

$$\mathcal{E}\ell\ell^*(\ ) = \mathcal{E}\ell\ell_* \underset{MU_*}{\otimes} MU^*(\ ) \cong \mathcal{E}\ell\ell_* \underset{E\ell\ell_*}{\otimes} E\ell\ell^*(\ ), \tag{1.5}$$

$$\mathcal{E}\ell\ell_*(\ ) = \mathcal{E}\ell\ell_* \underset{MU_*}{\otimes} MU_*(\ ) \cong \mathcal{E}\ell\ell_* \underset{E\ell\ell_*}{\otimes} E\ell\ell_*(\ ), \tag{1.6}$$

This ring $\mathcal{E}\ell\ell_*$ can be interpreted as a ring of *meromorphic modular forms with character in the finite cyclic group* $\operatorname{Hom}(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/12$. In other words, the usual modularity conditions on a holomorphic function $F\colon \mathcal{V} \longrightarrow \mathbb{C}$ are replaced by

$$F(\lambda \cdot \{\omega_1, \omega_2\}) = \lambda^{-k} F(\{\omega_1, \omega_2\}), \tag{1.7}$$

(1.8)

$$F(\{a\omega_1 + b\omega_2, c\omega_1 + d\omega_2\}) = \chi_F(A) F(\{\omega_1, \omega_2\}) \forall\, A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

for some character $\chi_F\colon \mathrm{SL}_2(\mathbb{Z}) \longrightarrow \mathbb{Q}/\mathbb{Z}$. Then $\Lambda = \eta^2$ is the square of Dedekind's η-function [25] and has character of order 12 which generates the finite cyclic group $\operatorname{Hom}(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$. Because of this, we may identify such a ring of 'twisted' modular forms with the extension $\mathcal{E}\ell\ell_*$ of $E\ell\ell_*$. Although we make no use of this here, there are advantages in having a 2-periodic cohomology theory rather than one of period 24.

2. Elliptic curves, Weierstrass formal group laws and complex orientations in elliptic cohomology

Given an analytic torus $\mathbb{C}/L$, we can construct a *Weierstrass cubic* (*elliptic curve*) (thought of as a projective cubic curve)

$$\mathfrak{C}_{\mathrm{W}}(L)\colon Y^2 Z = 4X^3 - \frac{1}{12} E_4(L) X Z^2 + \frac{1}{216} E_6(L) Z^3,$$

where the function E_{2n} is the $2n$th Eisenstein function of Section 1, regarded as a function of the lattice L. The classical theory of the Weierstrass function gives us an explicit uniformisation of this curve. We define an analytic isomorphism

$$\begin{aligned} \Phi\colon \mathbb{C}/L &\longrightarrow \mathfrak{C}_{\mathrm{W}}(L) \\ z + L &\longmapsto \begin{cases} [\wp(z, L), \wp'(z, L), 1], & \text{if } z \notin L, \\ [0, 1, 0], & \text{otherwise.} \end{cases} \end{aligned}$$

Here the Weierstrass function is normalised as in [6], so that for the lattice $L = 2\pi\mathrm{i}\,\langle \tau, 1\rangle$ with $\tau \in \mathfrak{H}$, we have

$$\wp(z, L) = \frac{1}{\left(e^{z/2} - e^{-z/2}\right)^2} + \sum_{n \geqslant 1} \left[\frac{q^n e^z}{(1 - q^n e^z)^2} + \frac{q^n e^{-z}}{\left(1 - q^n e^{-z}\right)^2} \right].$$

The local parameter

$$\mathcal{T}(z, L) = \frac{-2\wp(z, L)}{\wp'(z, L)}$$

is an elliptic function on $\mathfrak{C}_{\mathrm{W}}(L)$ which has a simple zero at each lattice point. The multiplication on $\mathfrak{C}_{\mathrm{W}}(L)$ gives rise to a formal group law

$$F_L^{E\ell\ell}(T_1, T_2) \in \mathbb{Z}[1/6][E_4(L), E_6(L)][[T_1, T_2]]$$

which we call the *Weierstrass formal group law associated to the lattice* L, and is determined by the relation

$$\mathcal{T}(z_1 + z_2, L) = F_L^{E\ell\ell}(\mathcal{T}(z_1, L), \mathcal{T}(z_2, L)).$$

Of course, the universal example for such formal group laws is furnished by the power series

$$F^{E\ell\ell}(T_1, T_2) \in \mathbb{Z}[1/6][Q, R, \Delta^{-1}][[T_1, T_2]] = E\ell\ell_*[[T_1, T_2]]$$

which is the canonical formal group law in elliptic cohomology. The natural choice of orientation for the canonical complex line bundle $\eta \longrightarrow \mathbb{C}\mathrm{P}^\infty$ then corresponds to $\mathcal{T} \in E\ell\ell_*[[\mathcal{T}]] \cong E\ell\ell^*(\mathbb{C}\mathrm{P}^\infty)$. See [6] for further details on these points. Evaluation of q-expansions gives rise to a homomorphism

$$E\ell\ell_* = \mathbb{Z}[1/6][Q, R, \Delta^{-1}] \longrightarrow KU[1/6]_*((q)) = \mathbb{Z}[1/6][t, t^{-1}]((q)),$$

in which we use the Bott generator $t \in KU_2$ to keep track of the weight which is half the topological grading. This an analogue of the classical Chern character, essentially discussed as such in [30], which focuses on modular forms of level 2 and uses the ring $KO[1/2]_*$. One major advantage to the use of level 2 modular forms and the original definition of elliptic cohomology is that the formal group law and its logarithm can be displayed more explicitly in terms of natural algebra generators of the coefficient ring; see [14] for some calculational observations.

3. The Hopf algebroid $E\ell\ell_* E\ell\ell$

In this section we will give some algebraic results on the cooperation algebra $E\ell\ell_* E\ell\ell = E\ell\ell_*(E\ell\ell)$. The construction of the functors $E\ell\ell^*(\ )$ and $E\ell\ell_*(\ )$ depends crucially on the following consequence of the Landweber Exact Functor Theorem [27] (the last statement follows from an argument similar to one for $E(n)$ in [31]).

Theorem 3.1. *There is an isomorphism of bimodules over* $E\ell\ell_*$

$$E\ell\ell_* E\ell\ell \cong E\ell\ell_* \underset{MU_*}{\otimes} MU_* MU \underset{MU_*}{\otimes} E\ell\ell_*$$

where we use the natural genus $MU_* \longrightarrow E\ell\ell_*$ *associated to the formal group law* $F^{E\ell\ell}$ *to form tensor products. Moreover,* $E\ell\ell_* E\ell\ell$ *is flat as both a left and right module over* $E\ell\ell_*$.

Corollary 3.2. *The pair* $(E\ell\ell_* E\ell\ell, E\ell\ell_*)$ *is a Hopf algebroid over* $\mathbb{Z}[1/6]$.
More generally, for any subring R *of* $\mathbb{Q}$ *containing* $\mathbb{Z}[1/6]$, *the pair*

$$(E\ell\ell_* E\ell\ell \underset{\mathbb{Z}[1/6]}{\otimes} R, E\ell\ell_* \underset{\mathbb{Z}[1/6]}{\otimes} R)$$

is a Hopf algebroid over R.

The term *Hopf algebroid* is thoroughly explained in [33]. The structure maps of $E\ell\ell_*E\ell\ell$ are derived ultimately from those of the 'universal' Hopf algebroid (MU_*MU, MU_*). Let $\eta_L, \eta_R \colon E\ell\ell_* \longrightarrow E\ell\ell_*E\ell\ell$ be the left and right units; we will often abuse notation and write $X = \eta_L(X)$.

Working over the rational numbers $\mathbb{Q}$ we have a simple description. First we note a consequence of the Landweber Exact Functor Theorem, which implies that multiplication by a prime p is a monomorphism on $E\ell\ell_*E\ell\ell$; this was also noted in [14] for example.

Proposition 3.3. *The rationalisation map $E\ell\ell_*E\ell\ell \longrightarrow E\ell\ell_*E\ell\ell \otimes \mathbb{Q}$ is injective.*

Proposition 3.4. *As graded $\mathbb{Q}$ algebras we have*

$$E\ell\ell_*E\ell\ell \otimes \mathbb{Q} = \mathbb{Q}[Q, R, \Delta^{-1}, \eta_R(Q), \eta_R(R), \eta_R(\Delta)^{-1}].$$

We also have a well known relationship between the two natural formal group laws over $E\ell\ell_*E\ell\ell$ and $E\ell\ell_*E\ell\ell \otimes \mathbb{Q}$. Let $\log^{E\ell\ell} T$ and $\log^{E\ell\ell'} T$ denote the logarithms of the images over $E\ell\ell_*E\ell\ell \otimes \mathbb{Q}$ of the canonical formal group law induced by η_L and η_R.

Proposition 3.5. *Let $B(T) = \sum_{k \geqslant 0} B_k T^{k+1}$ denote the strict isomorphism from the formal group law on $E\ell\ell_*E\ell\ell \otimes \mathbb{Q}$ induced from η_L to that induced from η_R. Then we have:*

(1) *as algebras over $E\ell\ell_* \otimes \mathbb{Q} = \eta_L(E\ell\ell_* \otimes \mathbb{Q})$,*

$$E\ell\ell_*E\ell\ell \otimes \mathbb{Q} = E\ell\ell_* \otimes \mathbb{Q}[\eta_R(Q), \eta_R(R), \eta_R(\Delta)^{-1}];$$

(2) $\log^{E\ell\ell} T = \log^{E\ell\ell'}(B(T))$;
(3) *for each $n \geqslant 0$, we have $B_n \in E\ell\ell_{2n}E\ell\ell$;*
(4) *as an $E\ell\ell_* = \eta_L(E\ell\ell_*)$ algebra, $E\ell\ell_*E\ell\ell$ is generated by the elements B_n with $n \geqslant 1$ together with $\eta_R(\Delta^{-1})$.*

We can describe $(E\ell\ell_*E\ell\ell, E\ell\ell_*)$ as a universal object.

Proposition 3.6. *Let R_* be any graded commutative ring, let F_1, F_2 be formal group laws over R_* induced from $E\ell\ell_*$ by the ring homomorphisms $\theta_1, \theta_2 \colon E\ell\ell_* \longrightarrow R_*$, and let $H \colon F_1 \cong F_2$ be a strict isomorphism over R_*. Then there is a unique ring homomorphism $\Theta \colon E\ell\ell_*E\ell\ell \longrightarrow R_*$ such that*

$$\Theta \circ \eta_L = \theta_1 \quad \textit{and} \quad \Theta \circ \eta_R = \theta_2$$

and the series $\Theta(B(X)) = \sum_{n \geqslant 0} \Theta(B_n) X^{n+1}$ satisfies

$$H(X) = \Theta(B(X)).$$

This follows from the analogous universality of (MU_*MU, MU_*).

4. Generalized modular forms

We continue to use the notation established in Section 1. Recall the left principal bundle

$$\begin{aligned} \mathcal{V} &\longrightarrow \mathcal{L}; \\ \{\omega_1, \omega_2\} &\longmapsto \langle \omega_1, \omega_2 \rangle \end{aligned}$$

with structure group $\mathrm{SL}_2(\mathbb{Z})$.

For any natural number $N > 0$, we denote by $\mathrm{M}_2(N)$ the set of 2×2 integer matrices with determinant N and set

$$(1/N)\mathrm{M}_2(N) = \{(1/N)A : A \in \mathrm{M}_2(N)\}.$$

Of course, these are isomorphic as right and left $\mathrm{SL}_2(\mathbb{Z})$ sets. The associated bundle

$$\pi_{\mathcal{V}(N)} \colon \mathcal{V}(N) = (1/N)M_2(N) \underset{\mathrm{SL}_2(\mathbb{Z})}{\times} \mathcal{V} \longrightarrow \mathcal{L}$$

has fibre $(1/N)\mathrm{M}_2(N)$. Given an oriented basis $\{\omega_1, \omega_2\}$ for a lattice L and $A \in \mathrm{M}_2(N)$ with

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we have an oriented basis

$$\left\{ \frac{a\omega_1 + b\omega_2}{N}, \frac{c\omega_1 + d\omega_2}{N} \right\}$$

for the lattice

$$L' = \left\langle \frac{a\omega_1 + b\omega_2}{N}, \frac{c\omega_1 + d\omega_2}{N} \right\rangle$$

which contains L with index N. Notice that each of the projection maps

$$\mathcal{V}(N) \xrightarrow{\pi_{\mathcal{V}(N)}} \mathcal{L}$$

is an infinite covering, with fibre isomorphic to the set $(1/N)\mathrm{M}_2(N) \cong \mathrm{M}_2(N)$.

Factoring out by the left action of any subgroup $G \leqslant \mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{V}(N)$, and we obtain a covering $\mathcal{V}(N) \longrightarrow G\backslash\mathcal{V}(N)$. If the subgroup G contains the congruence subgroup $\Gamma(N)$, then this is a finite covering. We will be particularly interested in the two extreme cases $G = \mathrm{SL}_2(\mathbb{Z})$ and $G = \Gamma(N)$. We set

$$\begin{aligned} \mathcal{L}(N) &= \mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{V}(N), \\ \mathcal{F}(N) &= \Gamma(N)\backslash\mathcal{V}(N), \end{aligned}$$

which admit finite covering maps

$$\begin{aligned} {}_{\mathcal{L}(N)}\pi \colon \mathcal{L}(N) &\longrightarrow \mathcal{L}, \\ {}_{\mathcal{F}(N)}\pi \colon \mathcal{F}(N) &\longrightarrow \mathcal{L}, \end{aligned}$$

whose fibres are the sets

$$\begin{aligned} \mathrm{SL}_2(\mathbb{Z})\backslash(1/N)\mathrm{M}_2(N) &\cong \mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{M}_2(N), \\ \Gamma(N)\backslash(1/N)\mathrm{M}_2(N) &\cong \Gamma(N)\backslash\mathrm{M}_2(N). \end{aligned}$$

Of course, these maps are holomorphic maps of complex analytic manifolds. The projection maps are also equivariant with respect to the obvious action of the complex units $\mathbb{C}^\times$ by multiplication.

The space $\mathcal{L}(N)$ can be viewed as the space of pairs of lattices $L \subseteq L'$ with index N. Similarly, we can interpret $\mathcal{F}(N)$ as the space of pairs $(L, \{\omega_1' + L, \omega_2' + L\})$, where

$$\omega_1' = \frac{a\omega_1 + b\omega_2}{N},$$
$$\omega_2' = \frac{c\omega_1 + d\omega_2}{N}$$

for an oriented basis $\{\omega_1, \omega_2\}$ of L and

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(N).$$

Thus $\{\omega_1', \omega_2'\}$ is an oriented basis of the module L'/L over the ring $\mathbb{Z}/N$. We will frequently make use of these interpretations without further comment.

Now we can make the following definition of a notion of level N modular forms.

Definition 4.1. Let $G \leqslant \mathrm{SL}_2(\mathbb{Z})$ be a subgroup containing $\Gamma(N)$. Then a holomorphic map

$$F \colon G\backslash\mathcal{V}(N) \longrightarrow \mathbb{C}$$

is a modular form of level N for G of weight k if for $\lambda \in \mathbb{C}^\times$,

$$F(G[(1/N)A, \{\lambda\omega_1, \lambda\omega_2\}]) = \lambda^{-k} F(G[(1/N)A, \{\omega_1, \omega_2\}]).$$

If $G = \Gamma(N)$, then we frequently refer to such a modular form as a modular form of level N.

Notice that for such a G and a subgroup G' containing $\Gamma(N)$, a modular form of weight k for G is also one for G'. Holomorphic functions $\mathcal{L}(N) \longrightarrow \mathbb{C}$ for which the composite

$$\Gamma(N)\backslash\mathcal{V}(N) \longrightarrow \mathcal{L}(N) \longrightarrow \mathbb{C}$$

is a modular form of level N will often be met in this work; we will loosely refer to these as level N modular forms on $\mathcal{L}(N)$.

Given such a modular form F of level N, we can evaluate F on the fibres over the lattices of the form $\langle \tau, 1\rangle$, where $\tau \in \mathfrak{H}$. For each pair (r, s) with $0 < r, s$ and $rs = N$, there is a function

$$f_{F,r,s} \colon \tau \longmapsto F\left(G\left[\begin{pmatrix} r/N & 0 \\ 0 & s/N \end{pmatrix}, \langle \tau, 1\rangle\right]\right),$$

with Fourier expansion of the form

$$\sum_{-\infty < n < \infty} a_n^{F,r,s} q^{n/N} \quad \text{where } q^{1/N} = e^{2\pi \mathrm{i}\tau/N}.$$

We will refer to these q-expansions as the *q-expansions of F along the fibres.*

For each coset $BG \in \mathrm{SL}_2(\mathbb{Z})/G$, we also have the holomorphic function

$$F_{|B}(G[(1/N)A, \{\omega_1, \omega_2\}]) = F(BGB^{-1}[(1/N)BA, \{\omega_1, \omega_2\}]).$$

Definition 4.2. The modular form F for G is holomorphic at infinity if for each coset $GB \in G\backslash \mathrm{SL}_2(\mathbb{Z})$ and (r,s) as above, the functions

$$\tau \longmapsto F_{|_B}\left(G\left[\begin{pmatrix} r/N & 0 \\ 0 & s/N \end{pmatrix}, \langle \tau, 1\rangle\right]\right)$$

have q-expansions

$$\sum_{-\infty<n<\infty} a_n^{F,r,s,B} q^{n/N}$$

with $a_n^{F,r,s,B} = 0$ for $n < 0$; similarly, it is meromorphic at infinity if its q-expansions have $a_n^{F,r,s,B} = 0$ for $n \ll 0$.

We will refer to the collection of q-expansions along the fibres of the functions $F_{|_B}$ as the *q-expansions of F at the cusps.*

Now let $K \subseteq \mathbb{C}$ be a subring which contains $1/6$, and let ζ_N be a primitive Nth root of 1.

Definition 4.3. The modular form F for G is *defined over the ring K* if all the q-expansion coefficients of all the functions $F_{|_B}$, with $BG \in \mathrm{SL}_2(\mathbb{Z})/G$, are in the ring $K[1/N, \zeta_N]$.

We now want to define a generalized modular form as a function on all of the spaces $\mathcal{L}(N)$ simultaneously in such a way that the restriction to each $\mathcal{L}(N)$ depends upon N in a controlled fashion. To do this we require that for each N we have a holomorphic function $F_N : \mathcal{L}(N) \longrightarrow \mathbb{C}$ which is simultaneously a modular form for each of the two lattices associated to each point of $\mathcal{L}$. Thus we will require that our function is induced from a suitable type of function upon the product space $\mathcal{L} \times \mathcal{L}$ via the product $\nu_{(N)}\pi\nu_{(N)}$ of the two projection maps to $\mathcal{L}$. Finally, we will do this uniformly by requiring that these functions $\mathcal{L} \times \mathcal{L} \longrightarrow \mathbb{C}$ are independent of N.

Remark 4.4. The following definitions may appear somewhat forced in that we need to work with certain proper subsets of $\mathrm{Map}(X \times Y, \mathbb{C})$. In fact, in the examples we consider, the spaces X and Y can be given the structures of *complex analytic spaces* X_h, Y_h as discussed in [38] and also more briefly in [17], Appendix B (in fact they are obtained as the analytic spaces associate to algebraic varieties over $\mathbb{C}$). Hence, we could characterise these sets of functions as analytic functions on the product $X_h \times Y_h$. The case of $\mathcal{L}$ itself follows since there is an analytic isomorphism between $\mathcal{L}$ and the affine variety

$$\left\{(x,y) \in \mathbb{C}^2 : x^3 - y^2 \neq 0\right\} \subseteq \mathbb{C}^2.$$

In order to avoid excessive technicalities, we proceed along the route below even though it may seem somewhat laboured to those well versed in algebraic geometry.

Recall that given two spaces X, Y, there is an embedding

$$\mathrm{Map}(X,\mathbb{C}) \underset{\mathbb{C}}{\otimes} \mathrm{Map}(Y,\mathbb{C}) \longrightarrow \mathrm{Map}(X \times Y, \mathbb{C}),$$

which sends the element $f \otimes g$ to the pointwise product function

$$f \cdot g : (x,y) \longmapsto f(x)g(y).$$

We will identify $\mathrm{Map}(X,\mathbb{C}) \otimes_{\mathbb{C}} \mathrm{Map}(Y,\mathbb{C})$ with its image in $\mathrm{Map}(X \times Y, \mathbb{C})$. More generally, given two vector subspaces $A \subseteq \mathrm{Map}(X,\mathbb{C})$ and $B \subseteq \mathrm{Map}(Y,\mathbb{C})$, we may identify the subspace $A \otimes_{\mathbb{C}} B \subseteq \mathrm{Map}(X, \mathbb{C}) \otimes_{\mathbb{C}} \mathrm{Map}(Y, \mathbb{C})$ with a subpace of $\mathrm{Map}(X \times Y, \mathbb{C})$.

Let $\mathrm{MF}(\mathbb{C})_k$ denote the set of all weight k modular forms, i.e., holomorphic functions $\mathcal{L} \longrightarrow \mathbb{C}$ satisfying the modularity condition

$$F(\lambda \cdot L) = \lambda^{-k} F(L) \quad \forall \lambda \in \mathbb{C}^{\times}.$$

Given a subring $K \subseteq \mathbb{C}$, let $\mathrm{MF}(K)_k$ denote the set of all modular forms whose associated q-expansions have coefficients in K.

We now make a series of definitions.

Definition 4.5. A modular form of weight k on $\mathcal{L} \times \mathcal{L}$ is a holomorphic map $F \colon \mathcal{L} \times \mathcal{L} \longrightarrow \mathbb{C}$ such that for $\lambda \in \mathbb{C}^{\times}$,

$$F(\lambda \cdot L_1, \lambda \cdot L_2) = \lambda^{-k} F(L_1, L_2),$$

and

$$F \in \sum_{r \in \mathbb{Z}} \mathrm{MF}(\mathbb{C})_r \underset{\mathbb{C}}{\otimes} \mathrm{MF}(\mathbb{C})_{k-r} \subseteq \mathrm{Map}(\mathcal{L} \times \mathcal{L}, \mathbb{C}).$$

We can now give our definition of a generalized modular form.

Definition 4.6. A generalized modular form of level 1 and weight k is the coproduct $F_{\bullet} = \coprod_{N \geqslant 1} F_N$ of a family of holomorphic maps of the form

$$F_N \colon \mathcal{L}(N) \xrightarrow{\nu(N)^{\pi} \times \pi_{\nu(N)}} \mathcal{L} \times \mathcal{L} \xrightarrow{F} \mathbb{C}$$

where $F \colon \mathcal{L} \times \mathcal{L} \longrightarrow \mathbb{C}$ is a fixed modular form of weight k on $\mathcal{L} \times \mathcal{L}$.

Notice that for each $N \geqslant 1$, F_N is a modular form of level N for $\mathrm{SL}_2(\mathbb{Z})$ of weight k.

Definition 4.7. The generalized modular form $F_{\bullet}$ is *defined over* K if for each $N \geqslant 1$, the modular form F_N of level N for $\mathrm{SL}_2(\mathbb{Z})$ is defined over K.

Definition 4.8. The generalized modular form $F_{\bullet}$ is holomorphic at infinity if for each $N \geqslant 1$, the modular form F_N of level N is holomorphic at ∞; similarly, $F_{\bullet}$ is meromorphic at infinity if each F_N is meromorphic at ∞.

Now let us consider the groups of all holomorphic (resp. meromorphic) generalized modular forms of weight k and defined over K, which we will denote by ${}^{\mathrm{Gen}}\mathrm{S}(K)_k$ (resp. ${}^{\mathrm{Gen}}\mathrm{M}(K)_k$). These can be combined into two graded rings ${}^{\mathrm{Gen}}\mathrm{S}(K)_*$ and ${}^{\mathrm{Gen}}\mathrm{M}(K)_*$ which are algebras over the rings $\mathrm{S}(K)_*$ and $\mathrm{M}(K)_*$ of Section 1. Since both of these rings are torsion free, we have

$$\begin{aligned} {}^{\mathrm{Gen}}\mathrm{S}(K)_* &\subseteq {}^{\mathrm{Gen}}\mathrm{S}(K\mathbb{Q})_* \\ {}^{\mathrm{Gen}}\mathrm{M}(K)_* &\subseteq {}^{\mathrm{Gen}}\mathrm{M}(K\mathbb{Q})_*, \end{aligned}$$

where $K\mathbb{Q}$ is the smallest subring of $\mathbb{C}$ containing both K and $\mathbb{Q}$. We can easily prove the next result.

Theorem 4.9. *As graded algebras over the rings* $\mathrm{S}(\mathbb{Q})_*$ *and* $\mathrm{M}(\mathbb{Q})_*$ *we have*

$$
\begin{aligned}
{}^{\mathrm{Gen}}\mathrm{S}(\mathbb{Q})_* &= \mathrm{S}(\mathbb{Q})_*[E_4', E_6'] \\
{}^{\mathrm{Gen}}\mathrm{M}(\mathbb{Q})_* &= \mathrm{M}(\mathbb{Q})_*[E_4', E_6', {\Delta'}^{-1}],
\end{aligned}
$$

where for each $N \geqslant 1$,

$$
\begin{aligned}
E_{2n}' &= E_{2n} \circ {}_{\mathcal{L}(N)}\pi, \\
\Delta' &= \Delta \circ {}_{\mathcal{L}(N)}\pi
\end{aligned}
$$

as functions $\mathcal{L}(N) \longrightarrow \mathbb{C}$.

Recall from the definition of elliptic cohomology that $\mathrm{M}(\mathbb{Z}[1/6])_* = E\ell\ell_*$. By Proposition 3.4, we obtain the following.

Corollary 4.10. *As graded algebras over* $E\ell\ell\mathbb{Q}_* \cong E\ell\ell_* \otimes \mathbb{Q}$,

$$
{}^{\mathrm{Gen}}\mathrm{M}(\mathbb{Q})_* \cong E\ell\ell\mathbb{Q}_* E\ell\ell \cong E\ell\ell_* E\ell\ell \otimes \mathbb{Q}.
$$

This suggests that we ought to be able to describe $E\ell\ell_* E\ell\ell$ in terms of the ring ${}^{\mathrm{Gen}}\mathrm{M}(\mathbb{Z}[1/6])_*$. The crucial question is of course what effect integrality conditions have on the existence of generalized modular forms. The complete algebraic calculations of ${}^{\mathrm{Gen}}\mathrm{S}(\mathbb{Z}[1/6])_*$ and ${}^{\mathrm{Gen}}\mathrm{M}(\mathbb{Z}[1/6])_*$ will be given later, using work of N. Katz [23].

We will now discuss a multiplicative structure on the space $\coprod_{n\geqslant 1} \mathcal{L}(N)$, which induces coproducts on the rings of generalized modular forms.

For $M, N \geqslant 1$, there is a partial product map

(4.1)
$$
(1/M)\mathrm{M}_2(M) \underset{\mathrm{SL}_2(\mathbb{Z})}{\times} \mathcal{V} \underset{\mathcal{L}}{\times} (1/N)\mathrm{M}_2(N) \underset{\mathrm{SL}_2(\mathbb{Z})}{\times} \mathcal{V} \longrightarrow (1/MN)\mathrm{M}_2(MN) \underset{\mathrm{SL}_2(\mathbb{Z})}{\times} \mathcal{V}
$$

which is defined on elements by the formula

$$
([A, \{\omega_1', \omega_2'\}], [B, \{\omega_1, \omega_2\}]) \longmapsto [ATB, \{\omega_1, \omega_2\}], \tag{4.2}
$$

where we have

$$
\{\omega_1', \omega_2'\} = TB\{\omega_1, \omega_2\}
$$

for some $T \in \mathrm{SL}_2(\mathbb{Z})$. Here the symbol $\underset{\mathcal{L}}{\times}$ implies that we form the pullback of the diagram

$$
(1/M)\mathrm{M}_2(M) \underset{\mathrm{SL}_2(\mathbb{Z})}{\times} \mathcal{V} \xrightarrow{\pi_{\mathcal{V}}} \mathcal{L} \xleftarrow{{}_{\mathcal{V}}\pi} (1/N)\mathrm{M}_2(N) \underset{\mathrm{SL}_2(\mathbb{Z})}{\times} \mathcal{V}.
$$

It is easily verified that this partial product is then compatible with the action of $\mathrm{SL}_2(\mathbb{Z})$ in the sense that it passes down to a partial product

$$
\mathcal{L}(M) \underset{\mathcal{L}}{\times} \mathcal{L}(N) \longrightarrow \mathcal{L}(MN).
$$

This product can be viewed as making the space

$$
\mathcal{L}^{\bullet} = \coprod_{N\geqslant 1} \mathcal{L}(N)
$$

into a 'monoid over $\mathcal{L}$'. It is clearly associative and the space $\mathcal{L}(1)$ acts via the identity. Taking functions on this space we obtain a coproduct which sends the function $F\colon \mathcal{L}^\bullet \longrightarrow \mathbb{C}$ to the function

$$\mathcal{L}^\bullet \underset{\mathcal{L}}{\times} \mathcal{L}^\bullet \longrightarrow \mathcal{L}^\bullet \xrightarrow{F} \mathbb{C}.$$

The space $\mathcal{L}^\bullet$ over $\mathcal{L}$ appears to play a rôle in elliptic cohomology analogous to that of the non-zero integers in K-theory, where they index (stable) Adams operations. We will make this more explicit in future work, but in this paper we will only demonstrate its connections with stable operations. In Section 5, we will describe the space $\mathcal{L}^\bullet$ in a more algebraic fashion.

5. Isogenies of elliptic curves and cooperation algebras

By an *elliptic curve* $\mathfrak{C}$ over the complex numbers $\mathbb{C}$ we will mean a non-singular Riemann surface of genus 1 with a distinguished basepoint $\mathrm{O}_{\mathfrak{C}}$. It is known that this can be *uniformised,* i.e., there is an analytic isomorphism

$$\Phi\colon \mathfrak{C} \cong \mathbb{C}/L$$
$$\Phi(\mathrm{O}_{\mathfrak{C}}) = 0 + L,$$

where $L \subseteq \mathbb{C}$ is a lattice. Particular examples are furnished by the Weierstrass cubics of Section 5. Moreover, the torus $\mathbb{C}/L$ is unique up to an analytic isomorphism of the form

$$[\lambda]\colon \mathbb{C}/L \longrightarrow \mathbb{C}/L'$$

where $[\lambda]$ is induced by multiplication by λ and $\lambda \cdot L = L'$. We can scale L so that it has the form $L = \langle \tau, 1\rangle$ for some $\tau \in \mathfrak{H}$ (the upper half plane); then Φ is unique up to analytic automorphism of $\mathbb{C}/L$. Of course, there is a canonical abelian group structure on $\mathbb{C}/L$ which is transferred to $\mathfrak{C}$ by Φ, and $\mathfrak{C}$ is an analytic group with $\mathrm{O}_{\mathfrak{C}}$ as its zero element.

Given two elliptic curves $\mathfrak{C}_1$, $\mathfrak{C}_2$ over $\mathbb{C}$, an *isogeny* from $\mathfrak{C}_1$ to $\mathfrak{C}_2$ is an analytic homomorphism of groups $\Theta\colon \mathfrak{C}_1 \longrightarrow \mathfrak{C}_2$ such that $\ker\Theta$ is finite (it is then necessarily surjective). Let $\deg\Theta = |\ker\Theta|$, the *degree* of Θ, and $K_\Theta \subseteq \mathbb{C}$ be the unique lattice such that $K_\Theta/L_1 = \ker\Theta$. For $\mathfrak{C}_1 = \mathbb{C}/L_1$ and $\mathfrak{C}_2 = \mathbb{C}/L_2$ such an isogeny has to be of the form $[\lambda]$ with

$$\lambda \cdot K_\Theta = L_2,$$

and thus there is a unique factorisation

$$\mathbb{C}/L_1 \longrightarrow \mathbb{C}/K_\Theta \xrightarrow{[\lambda]} \mathbb{C}/L_2 \tag{5.1}$$

where the first map is induced by the canonical inclusion $L_1 \longrightarrow K_\Theta$. We will say that an isogeny is *strict* if $\lambda = 1$. Notice that for a strict isogeny,

$$L_1 \subseteq L_2$$

has finite index and also

$$L_2/L_1 = \ker([1]\colon \mathbb{C}/L_1 \longrightarrow \mathbb{C}/L_2).$$

From the above discussion, we see that the category of elliptic curves over $\mathbb{C}$ with isogenies as morphisms, is naturally equivalent to the category of tori $\mathbb{C}/L$

and isogenies, which will denote by $\mathbf{Iso}_{\mathbb{C}}$. We will restrict attention to elliptic curves of the form $\mathbb{C}/L$ and work with the category $\mathbf{SIso}_{\mathbb{C}}$ of all such elliptic curves together with their strict isogenies as morphisms.

We can decompose these categories $\mathbf{Iso}_{\mathbb{C}}$ and $\mathbf{SIso}_{\mathbb{C}}$ into disjoint unions

$$\mathbf{Iso}_{\mathbb{C}} = \coprod_{N\geqslant 1} \mathbf{Iso}_{\mathbb{C}}(N)$$
$$\mathbf{SIso}_{\mathbb{C}} = \coprod_{N\geqslant 1} \mathbf{SIso}_{\mathbb{C}}(N)$$

where $\mathbf{Iso}_{\mathbb{C}}(N)$ consists of isogenies with degree N and we have the equation $\mathbf{SIso}_{\mathbb{C}}(N) = \mathbf{Iso}_{\mathbb{C}}(N) \cap \mathbf{SIso}_{\mathbb{C}}$. Of course, the set $\mathbf{SIso}_{\mathbb{C}}(1)$ can be viewed as the set of objects in the categories $\mathbf{Iso}_{\mathbb{C}}$ and $\mathbf{SIso}_{\mathbb{C}}$.

We can identify the morphism sets $\mathbf{SIso}_{\mathbb{C}}(N)$ with the underlying sets of the spaces $\mathcal{L}(N)$ defined in Section 4, since by construction a point of $\mathcal{L}(N)$ is equivalent to an inclusion of lattices $L \subseteq L'$ of index N. Moreover, the two projections $\pi_{\mathcal{L}(N)},\, {}_{\mathcal{L}(N)}\pi \colon \mathcal{L}(N) \longrightarrow \mathcal{L}$ simply pick out these two lattices, which are the domain and codomain of a unique morphism in $\mathbf{SIso}_{\mathbb{C}}(N) \subseteq \mathbf{SIso}_{\mathbb{C}}$. Thus, we have the following result.

Proposition 5.1. *There is an isomorphism of small categories*

$$\mathbf{SIso}_{\mathbb{C}} \cong \mathcal{L}^{\bullet}$$

under which

$$\mathbf{SIso}_{\mathbb{C}}(N) \cong \mathcal{L}(N)$$

for each $N \geqslant 1$. The category $\mathbf{SIso}_{\mathbb{C}}$ is therefore naturally topologised and is the union of countably infinitely many complex manifolds $\mathbf{SIso}_{\mathbb{C}}(N)$.

This result together with the ideas of Section 4 gives us an interesting class of functions on $\mathbf{SIso}_{\mathbb{C}}$, which are analytic when restricted to the spaces $\mathbf{SIso}_{\mathbb{C}}(N) \cong \mathcal{L}(N)$. We will freely interpret generalized modular forms as functions on the category $\mathbf{SIso}_{\mathbb{C}}$. Of course, the structure maps of the category $\mathbf{SIso}_{\mathbb{C}}$ correspond to the partial monoid structure on $\mathcal{L}^{\bullet}$; thus there will be a coproduct structure on the ring of generalized modular forms. This structure becomes interesting when we tensor up with a subring $R \subseteq \mathbb{Q}$ and force morphisms to become invertible; we then obtain the structure of a Hopf algebroid on an appropriate ring of generalized modular forms.

Now most of the morphisms in $\mathbf{Iso}_{\mathbb{C}}$ and $\mathbf{SIso}_{\mathbb{C}}$ are not invertible and we will need to form various categories of fractions for these. Let $R \subseteq \mathbb{Q}$ be a subring of the rational numbers and let $R^{\times}_{+}$ denote the subgroup $R^{\times} \cap \mathbb{Q}_{+}$ of all positive units in R. We wish to invert the strict isogenies $[1] \colon \mathbb{C}/L_1 \longrightarrow \mathbb{C}/L_2$ with $|L_2/L_1| \in R^{\times}_{+}$. To do this we replace the $\mathbb{Z}$ lattices L_1 and L_2 by the R lattices $RL_1 \cong R \otimes_{\mathbb{Z}} L_1$ and $RL_2 \cong R \otimes_{\mathbb{Z}} L_2$, and consider 'isogenies' of the form

$$[u] \colon \mathbb{C}/RL_1 \longrightarrow \mathbb{C}/RL_2$$

where $u \in R^{\times}_{+}$. Notice that such an isogeny has trivial kernel, and has inverse

$$[u^{-1}] \colon \mathbb{C}/RL_2 \longrightarrow \mathbb{C}/RL_1 .$$

Such morphisms lie in a category $\mathbf{Iso}_{\mathbb{C}}^{R_+^\times}$ whose objects are those of $\mathbf{Iso}_{\mathbb{C}}$ and where for any two lattices L_1 and L_2 for which $RL_1 = RL_2$, there is unique morphism $[u]\colon \mathbb{C}/L_1 \longrightarrow \mathbb{C}/L_2$ whenever $u \in R_+^\times$. We will call such a morphism an *R-isogeny;* furthermore, if $u = 1$, then we say that it is a *strict R-isogeny.* The strict R-isogenies form a subcategory $\mathbf{SIso}_{\mathbb{C}}^{R_+^\times}$ of $\mathbf{Iso}_{\mathbb{C}}^{R_+^\times}$. If two lattices L_1 and L_2 satisfy $RL_1 = RL_2$, then we will say that they are *R-commensurable.* It is easy to see that the notion of being R-commensurable is an equivalence relation. Notice that if L_1 and L_2 are R-commensurable, then the lattice $L_1 \cap L_2$ is R-commensurable with both L_1 and L_2; moreover, the unique diagram

$$\mathbb{C}/L_1 \longleftarrow \mathbb{C}/L_1 \cap L_2 \longrightarrow \mathbb{C}/L_2$$

in $\mathbf{SIso}_{\mathbb{C}}$ gives rise to a unique diagram

$$\mathbb{C}/L_1 \longrightarrow \mathbb{C}/L_1 \cap L_2 \longrightarrow \mathbb{C}/L_2$$

in $\mathbf{SIso}_{\mathbb{C}}^{R_+^\times}$.

Theorem 5.2. *The functor $\mathbf{SIso}_{\mathbb{C}} \longrightarrow \mathbf{SIso}_{\mathbb{C}}^{R_+^\times}$ which is the identity on objects and sends the strict isogeny $\mathbb{C}/L_1 \longrightarrow \mathbb{C}/L_2$ to the strict R isogeny $[1]\colon \mathbb{C}/L_1 \longrightarrow \mathbb{C}/L_2$ is the localization of $\mathbf{SIso}_{\mathbb{C}}$ with respect to all morphisms $[1]\colon \mathbb{C}/L' \longrightarrow \mathbb{C}/L'$ for which $|\ker[1]| \in R_+^\times$.*

Notice that in particular this means that a strict isogeny $\mathbb{C}/L \longrightarrow \mathbb{C}/(1/N)L$ for which $N \in R_+^\times \cap \mathbb{N}$ always has an inverse in $\mathbf{SIso}_{\mathbb{C}}^{R_+^\times}$.

In practise, we will work with rings R for which $\mathbb{Z}[1/6] \subseteq R$, although this restriction is only important when we consider rings of modular forms as rings of functions on $\mathbf{Iso}_{\mathbb{C}}$ and $\mathbf{Iso}_{\mathbb{C}}^{R_+^\times}$.

We end this section by introducing another collection of categories. Let $\mathbf{SL\,Iso}_{\mathbb{C}}^{R_+^\times}$ denote the category whose objects are lattices $L \in \mathcal{L}$, and where whenever $RL_1 = RL_2$, the morphisms from L_1 to L_2 are the orientation preserving monomorphisms $L_1 \longrightarrow L_2$ which induce R-linear isomorphisms $RL_1 \cong RL_2$. In particular, when $R = \mathbb{Z}$, there are morphisms $L_1 \longrightarrow L_2$ if and only if $L_1 = L_2$; on the other hand, when $R = \mathbb{Q}$, there are morphisms $L_1 \longrightarrow L_2$ if and only if $\mathbb{Q}L_1 = \mathbb{Q}L_2$. In the case $R = \mathbb{Z}$, we may identify $\mathbf{SL\,Iso}_{\mathbb{C}} = \mathbf{SL\,Iso}_{\mathbb{C}}^{R_+^\times}$ with the space

$$\mathcal{V}^\bullet = \coprod_{N \geqslant 1} \mathcal{V}(N).$$

6. The action of isogenies on Weierstrass formal groups and operations in elliptic cohomology

Given a strict isogeny $[1]\colon \mathbb{C}/L_1 \longrightarrow \mathbb{C}/L_2$ of degree N, together with a modular form F of level 1, the function

$$([1]\colon \mathbb{C}/L_1 \longrightarrow \mathbb{C}/L_2) \longmapsto F(L_2)$$

is a modular form in the variable L_2. If we choose an oriented basis for L_1 and use this to make the identifications

$$\mathrm{SL}(L_1) \cong \mathrm{SL}_2(\mathbb{Z}) \tag{6.1}$$

and

$$\mathrm{SL}((1/N)L_1) \cong \mathrm{SL}_2(\mathbb{Z}), \tag{6.2}$$

then we can interpret this function as a modular form for the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ corresponding to

$$\mathrm{SL}(L_1) \cap \mathrm{SL}(L_2) \subseteq \mathrm{SL}((1/N)L_1) \cong \mathrm{SL}_2(\mathbb{Z})$$

under the isomorphism of (6.2). The proof of the following key result is similar to arguments of [6, 9].

Proposition 6.1. *The formal group laws $F^{E\ell\ell}_{L_1}$ and $F^{E\ell\ell}_{L_2}$ are strictly isomorphic over the ring of level N modular forms on $\mathcal{L}(N)$ defined over the ring $\mathbb{Z}[1/6]$.*

Proof. The coefficients of $F^{E\ell\ell}_{L_1}$ and $F^{E\ell\ell}_{L_2}$ considered as functions of the pair $L_1 \subseteq L_2$ are level N modular forms. In fact they lie in the rational subalgebra $\mathbb{Q}[E_4(L_1), E_6(L_1), E_4(L_2), E_6(L_2)] \subseteq \mathbb{C}$ generated by the complex numbers $E_r(L_s)$. The series $\mathcal{T}(X, L_1)$ and $\mathcal{T}(X, L_2)$ provide strict isomorphisms from the additive group law to $F^{E\ell\ell}_{L_1}$ and $F^{E\ell\ell}_{L_2}$, hence there is a strict isomorphism

$$\varphi_{L_1,L_2} \colon F^{E\ell\ell}_{L_1} \longrightarrow F^{E\ell\ell}_{L_2}$$

with coefficients in the latter ring. Now by specialising to the case where $L_1 = \langle \tau, 1\rangle$ ($\tau \in \mathfrak{H}$) the series $\varphi_{L_1,L_2}(X)$ gives a q-expansion

$$\varphi_{\langle \tau,1\rangle,L_2}(X) \in \mathbb{Q}[\zeta_N][[q^{1/N}]][[X]].$$

Following [6], we can use the theory of Tate curves described in [22] to deduce that the coefficients of $F^{E\ell\ell}_{\langle \tau,1\rangle}$ and $F^{E\ell\ell}_{L_2}$ actually lie in the rings $\mathbb{Z}[1/6][[q]]$ and $\mathbb{Z}[1/6N, \zeta_N][[q^{1/N}]]$. Hence

$$\varphi_{\langle \tau,1\rangle,L_2}(X) \in \mathbb{Z}[1/6N, \zeta_N][[q^{1/N}]][[X]],$$

showing that the coefficients of $\varphi_{L_1,L_2}(X)$ are level N modular forms on $\mathcal{L}(N)$ defined over $\mathbb{Z}[1/6N]$. $\square$

Let $\varphi_{L_1,L_2}(X)$ be the unique strict isomorphism from $F^{E\ell\ell}_{L_1}$ to $F^{E\ell\ell}_{L_2}$ used in the proof of this result; we will write φ when the isogeny is understood. The following Corollary makes use of the fact that the considerations of the above proof are essentially independent of $L \subseteq L'$. Indeed, the coefficients of $\varphi_{L_1,L_2}(X)$ are rational polynomials in the coefficients of the formal group laws $F^{E\ell\ell}_{L_1}$ and $F^{E\ell\ell}_{L_2}$, independently of the lattices $L_1 \subseteq L_2$ and the index N.

Corollary 6.2. *The coefficient of X^{n+1} in $\varphi_{L_1,L_2}(X)$ when considered as a function of pairs $L_1 \subseteq L_2$ for arbitrary $N \geqslant 1$, is a holomorphic generalized modular form of weight n, i.e., is contained in ${}^{\mathrm{Gen}}\mathrm{S}(\mathbb{Z}[1/6])_n \subseteq {}^{\mathrm{Gen}}\mathrm{M}(\mathbb{Z}[1/6])_n$.*

Now given any R isogeny $[u]\colon \mathbb{C}/L_1 \longrightarrow \mathbb{C}/L_2$, we can assume that $L_2 \subseteq (1/N)L_1$ for some $N \in R_+^\times$ and then an easy calculation gives

$$\begin{aligned} \mathcal{T}(uz, L_2) &= [u]_{F^{E\ell\ell}_{L_2}}(\mathcal{T}(z, L_2)) \\ &= \varphi\left([u]_{F^{E\ell\ell}_{(1/N)L_1}}\mathcal{T}(z,(1/N)L_1)\right) \\ &= \varphi\left((1/N)[uN]_{F^{E\ell\ell}_{L_1}}\mathcal{T}(z,L_1)\right). \end{aligned} \tag{6.3}$$

But this is a power series in $\mathcal{T}(z, L_1)$ with coefficients in the ring of level N modular forms on $\mathcal{L}(N)$ defined over R. Hence, any strict isogeny $[u]$ as above induces an isomorphism between the formal group law associated with the elliptic curve $\mathbb{C}/L_1$ and a 'twisted version' of that associated to $\mathbb{C}/L_2$. In the case where $[u] = [1]$ is strict, so is the induced isomorphism of formal group laws. Notice that this implies that for each strict R-isogeny $[1]\colon \mathbb{C}/L_1 \longrightarrow L_2$, there is a ring homomorphism Ψ_{L_1,L_2} with domain MU_*MU and extending the two homomorphisms

$$\begin{aligned} MU_* &\xrightarrow{\varphi_{L_1}} R[1/6][E_4(L_1), E_6(L_1)], \\ MU_* &\xrightarrow{\varphi_{L_2}} R[1/6][E_4(L_2), E_6(L_2)] \end{aligned}$$

which classify the formal group laws $F^{E\ell\ell}_{L_1}$ and $F^{E\ell\ell}_{L_2}$; this takes values which are level N modular forms when considered as functions of $L_1 \subseteq L_2$.

It is now immediate that there is a unique homomorphism

$$E\ell\ell_* \underset{MU_*}{\otimes} MU_*MU \underset{MU_*}{\otimes} E\ell\ell_* \longrightarrow E\ell\ell_*E\ell\ell \longrightarrow {}^{\mathrm{Gen}}\mathrm{M}(R)_*$$

which specialises for each pair $L_1 \subseteq L_2$ to give Ψ_{L_1,L_2}, using Corollary 6.2. In the case of a strict R-isogeny of the form $[1]\colon L \longrightarrow (1/N)L$, we find that the left unit on an element $F \in E\ell\ell_{2n}$ is sent to N^nF by this homomorphism; in this case we can produce a multiplicative stable operation in elliptic cohomology:

$$\begin{aligned} \psi^N\colon E\ell\ell^*(\ ) \cong (S^0 \wedge E\ell\ell)^*(\ ) &\longrightarrow (E\ell\ell \wedge E\ell\ell)^*(\ ) \\ &\xrightarrow{\cong} E\ell\ell_*E\ell\ell \underset{E\ell\ell_*}{\otimes} E\ell\ell^*(\ ) \\ &\xrightarrow{\Psi_{L_1,L_2}} E\ell\ell R_* \underset{E\ell\ell_*}{\otimes} E\ell\ell^*(\ ) \\ &\longrightarrow E\ell\ell R^*(\ ), \end{aligned} \tag{6.4}$$

which makes use of the above homomorphism $E\ell\ell_*E\ell\ell R \longrightarrow E\ell\ell R_*$. This is the Adams operation ψ^N mentioned in [6], and has a unique extension to a stable operation

$$\psi^N\colon E\ell\ell R^*(\ ) \longrightarrow E\ell\ell R^*(\ ).$$

For a fixed L and $N \in R_+^\times$, we can take all of the induced ring homomorphisms $E\ell\ell_* \longrightarrow E\ell\ell R_*$ and average them (i.e., sum up and divide by N). This gives rise to a left $E\ell\ell_*$-linear homomorphism

$$\widetilde{\mathrm{T}}\colon E\ell\ell_*E\ell\ell \longrightarrow E\ell\ell R_*$$

which yields a stable operation

$$\text{(6.5)}\qquad \overline{\mathrm{T}}_N\colon E\ell\ell^*(\ ) \cong (S^0\wedge E\ell\ell)^*(\ ) \cong E\ell\ell_*E\ell\ell \underset{E\ell\ell_*}{\otimes} E\ell\ell^*(\ )$$
$$\xrightarrow{\widetilde{\mathrm{T}}} E\ell\ell R^* \underset{E\ell\ell_*}{\otimes} E\ell\ell^*(\ )$$
$$\cong E\ell\ell R^*(\ )$$

that is merely additive; again there is a unique extension to a stable operation $\overline{\mathrm{T}}_N\colon E\ell\ell R^*(\ ) \longrightarrow E\ell\ell R^*(\ )$. This is the extension of the Nth Hecke operator constructed in [6]. This type of operation requires that we use not just the ring $E\ell\ell R_*$ but the larger ring of modular forms of level N to build enough multiplicative operations over which we symmetrise to get an operation within the theory $E\ell\ell R^*(\ )$ itself. This sort of consideration is not necessary in K-theory, and represents a considerable complication in understanding the operations in elliptic cohomology.

Of course, the above discussion can also be interpreted in the light of the observation in Section 5 that the rings of generalized modular forms may be viewed as functions on the categories $\mathbf{Iso}_{\mathbb{C}}^{R^\times_+}$ and $\mathbf{SIso}_{\mathbb{C}}^{R^\times_+}$. Indeed, given an R-isogeny $[u]\colon \mathbb{C}/L_1 \longrightarrow \mathbb{C}/L_2$, the coefficients of the power series discussed above can be viewed as functions on the category $\mathbf{Iso}_{\mathbb{C}}^{R^\times_+}$ and hence as elements of ${}^{\mathrm{Gen}}\mathrm{S}(\mathbb{C})_* \subseteq {}^{\mathrm{Gen}}\mathrm{M}(\mathbb{C})_*$. A careful consideration of q-expansions actually shows that they lie in ${}^{\mathrm{Gen}}\mathrm{S}(R)_* \subseteq {}^{\mathrm{Gen}}\mathrm{M}(R)_*$ provided that we make the standard assumption that $\mathbb{Z}[1/6] \subseteq R$. This provides us with a natural homomorphism $E\ell\ell_*E\ell\ell \longrightarrow {}^{\mathrm{Gen}}\mathrm{M}(R)_*$. Later we will demonstrate the following theorem.

Theorem 6.3. *For each subring $R \subseteq \mathbb{Q}$ containing $1/6$, there is an isomorphism of graded rings*

$$E\ell\ell_*E\ell\ell R \cong E\ell\ell_*E\ell\ell \otimes R \longrightarrow {}^{\mathrm{Gen}}\mathrm{M}(R)_*,$$

and moreover this is an isomorphism of Hopf algebroids over R.

The antipode in ${}^{\mathrm{Gen}}\mathrm{M}(R)_*$ is induced by the inverse map in the category $\mathbf{Iso}_{\mathbb{C}}^{R^\times_+}$, and corresponds under this isomorphism to the antipode in $E\ell\ell_*E\ell\ell R$.

7. Some rings of numerical Laurent polynomials and K-theory cooperations

In this section we review the properties of some rings of *numerical (Laurent) polynomials* in sufficient detail for our purposes in calculating the rings of generalized modular forms contained in Section 9. The present section owes much to previous joint work with Francis Clarke, see [11] and [4]; for more on the topological connections, see [3, 2].

Let $K \subseteq \mathbb{Q}$ be a subring. Then we define the ring of *numerical polynomials over K* to be

$$\mathrm{A}(w;K) = \{f(w) \in \mathbb{Q}[w] : f(r) \in K\ \forall r \in \mathbb{Z}\}.$$

Similarly, we define the ring of *stably numerical (Laurent) polynomials over K* to be

$$\mathrm{A}^{\mathrm{S}}(w;K) = \{f(w) \in \mathbb{Q}[w,w^{-1}] : f(r) \in K[1/r]\ \forall r \in \mathbb{Z},\ 0 \neq r\}.$$

Finally we define the subring of *semistable numerical polynomials over K* by

$$\mathrm{A}_0^{\mathrm{S}}(w;K)=\mathrm{A}^{\mathrm{S}}(w;K)\cap\mathbb{Q}[w].$$

We set $\mathrm{A}(w)=\mathrm{A}(w;\mathbb{Z})$, $\mathrm{A}^{\mathrm{S}}(w)=\mathrm{A}^{\mathrm{S}}(w;\mathbb{Z})$ and $\mathrm{A}_0^{\mathrm{S}}(w)=\mathrm{A}_0^{\mathrm{S}}(w;\mathbb{Z})$.

Proposition 7.1. *As a module over K, $\mathrm{A}(w;K)$ has a basis consisting of the binomial coefficient polynomials $\mathrm{C}_n(w)=\binom{w}{n}$ for $n\geqslant 0$. Hence we have an isomorphism of algebras over K,*

$$\mathrm{A}(w;K)\cong\mathrm{A}(w)\underset{\mathbb{Z}}{\otimes}K.$$

As algebras over K,

$$\mathrm{A}^{\mathrm{S}}(w;K)=\mathrm{A}(w;K)[w^{-1}].$$

Proofs of these results are given in [11].

Let us now assume that $K=\mathbb{Z}_{(p)}$, the ring of p-local integers for a prime p. Let $\mathrm{ord}_p(h(w))$ be the minimum value of ord_p on the coefficients of a Laurent polynomial $h(w)$, or equivalently

$$\mathrm{ord}_p(h(w))=\min\{\mathrm{ord}_p(h(a)):a\in\mathbb{Z}_{(p)}^{\times}\}.$$

We define increasing filtrations on $\mathrm{A}^{\mathrm{S}}(w;\mathbb{Z}_{(p)})$ and $\mathrm{A}_0^{\mathrm{S}}(w;\mathbb{Z}_{(p)})$ as follows. Let

$$\begin{aligned}M^k&=\{f(w)\in\mathrm{A}^{\mathrm{S}}(w;\mathbb{Z}_{(p)}):p^kf(w)\in\mathbb{Z}_{(p)}[w,w^{-1}]\},\\ M_0^k&=\{f(w)\in\mathrm{A}_0^{\mathrm{S}}(w;\mathbb{Z}_{(p)}):p^kf(w)\in\mathbb{Z}_{(p)}[w]\}=M^k\cap\mathrm{A}_0^{\mathrm{S}}(w;\mathbb{Z}_{(p)}).\end{aligned}$$

Clearly we have $M^0=\mathbb{Z}_{(p)}[w,w^{-1}]$ and $M_0^0=\mathbb{Z}_{(p)}[w]$; also the two filtrations

$$\begin{aligned}M^0\subseteq M^1\subseteq\cdots\subseteq M^k\subseteq\cdots\subseteq M^\infty&=\mathrm{A}^{\mathrm{S}}(w;\mathbb{Z}_{(p)})\\ M_0^0\subseteq M_0^1\subseteq\cdots\subseteq M_0^k\subseteq\cdots\subseteq M_0^\infty&=\mathrm{A}_0^{\mathrm{S}}(w;\mathbb{Z}_{(p)})\end{aligned}$$

are exhaustive. Let us investigate the successive quotients M^k/M^{k-1} and M_0^k/M_0^{k-1} for $k\geqslant 1$.

By Proposition (7.1), any element $f(w)\in\mathrm{A}^{\mathrm{S}}(w;\mathbb{Z}_{(p)})$ has the form

$$f(w)=\sum_{0\leqslant i\leqslant\mathrm{d}(f)}h_i(w)\mathrm{C}_i(w)\tag{7.1}$$

where $h_i(w)\in\mathbb{Z}_{(p)}[w,w^{-1}]$ and we assume that $h_{\mathrm{d}(f)}(w)\neq 0$. The p-adic ordinal of $n!$ is given by

$$\mathrm{ord}_p(n!)=\frac{n-\alpha_p(n)}{p-1},\tag{7.2}$$

where $\alpha_p(n)$ is the sum of the p-adic digits of n. In particular,

$$\mathrm{ord}_p(p^r!)=\frac{p^r-1}{p-1}=1+p+\cdots+p^{r-1}.\tag{7.3}$$

Now $\mathrm{C}_n(w)$ represents non-zero elements in the quotients

$$M^{\mathrm{ord}_p(n!)}/M^{\mathrm{ord}_p(n!)-1}\quad\text{and}\quad M_0^{\mathrm{ord}_p(n!)}/M_0^{\mathrm{ord}_p(n!)-1}.$$

Thus for a general element $f(w)$, we see that $f(w) \in M^k$ if and only if

$$k \geqslant \max\{\mathrm{ord}_p(n!) - \mathrm{ord}_p(h_n(w)) : 0 \leqslant n \leqslant \mathrm{d}(f)\}$$

and moreover it represents a non-zero element in M^k/M^{k-1} if and only if the last inequality is actually an equality.

It will be convenient to use a different basis for the p-local numerical polynomial ring $\mathrm{A}(w;\mathbb{Z}_{(p)})$. We require the following results taken from [4].

Proposition 7.2. *Define the following sequence of polynomials in* $\mathbb{Q}[w]$*:*

$$\begin{aligned}
\theta_0(w) &= w,\\
\theta_1(w) &= \frac{(\theta_0(w) - \theta_0(w)^p)}{p},\\
\theta_2(w) &= \frac{(\theta_0(w) - p\theta_1(w)^p - \theta_0(w)^{p^2})}{p^2},\\
&\ \vdots\\
\theta_r(w) &= \frac{\theta_0(w) - p^{r-1}\theta_{r-1}(w)^p - p^{r-2}\theta_{r-2}(w)^{p^2} - \cdots - \theta_0(w)^{p^r}}{p^r},\\
&\ \vdots
\end{aligned}$$

Then

(1) *for each* $r \geqslant 0$, $\theta_r(w) \in \mathrm{A}(w;\mathbb{Z}_{(p)})$ *and moreover defines a function* $\theta_r : \mathbb{Z}_{(p)} \to \mathbb{Z}_{(p)}$*;*

(2) *we have*

$$\mathrm{ord}_p(\theta_r(w)) = \frac{p^r - 1}{p - 1} = 1 + p + p^2 + \cdots + p^{r-1};$$

(3) *the monomials*

$$\theta_0(w)^{s_0}\theta_1(w)^{s_1}\cdots\theta_d(w)^{s_d} \quad \textit{for } 0 \leqslant s_r < p$$

form a $\mathbb{Z}_{(p)}$*-basis for* $\mathrm{A}(w;\mathbb{Z}_{(p)})$*;*

(4) *the monomials*

$$\theta_1(w)^{s_1}\cdots\theta_d(w)^{s_d} \quad \textit{for } 0 \leqslant s_r < p$$

span $\mathrm{A}^{\mathrm{S}}(w;\mathbb{Z}_{(p)})$ *as a module over* $\mathbb{Z}_{(p)}[w,w^{-1}]$*;*

(5) *for each* $k \geqslant 1$*, the monomials*

$$\theta_0(w)^{s_0}\theta_1(w)^{s_1}\cdots\theta_d(w)^{s_d} \quad \textit{for } 0 \leqslant s_r < p \textit{ and } 0 \leqslant s_0 < p-1$$

form a $\mathbb{Z}/p^k$*-basis for* $\mathrm{A}^{\mathrm{S}}(w;\mathbb{Z}_{(p)})/(p^k)$*, which can also be identified with the ring of functions* $\mathbb{Z}_{(p)}^{\times} \to \mathbb{Z}/p^k$ *which are continuous with respect the* p*-adic norm on the domain and the discrete topology on the range.*

We will set $\overline{\theta}_r(w) = w^{-1}\theta_r(w) \in \mathrm{A}_0^{\mathrm{S}}(w;\mathbb{Z}_{(p)})$ for $r \geqslant 1$.

Now consider an element of $\mathrm{A}_0^{\mathrm{S}}(w;\mathbb{Z}_{(p)})$ of the form

$$f(w) = c\theta_0(w)^{s_0}\overline{\theta}_1(w)^{s_1}\cdots\overline{\theta}_d(w)^{s_d}$$

where $0 \leqslant s_r < p$, $0 \leqslant s_0 < p-1$ and $c \in \mathbb{Z}_{(p)}$. Then the p-adic ordinal of this polynomial is

$$\mathrm{ord}_p(f(w)) = \mathrm{ord}_p(c) - \sum_{1 \leqslant j \leqslant d} s_j \frac{(p^j - 1)}{(p-1)},$$

and so $f(w) \in M_0^k$ if and only if

$$k \geqslant \sum_{1 \leqslant j \leqslant d} s_j \frac{(p^j - 1)}{(p-1)} - \mathrm{ord}_p(c),$$

and represents a non-zero element of M_0^k / M_0^{k-1} if and only if this is actually an equality.

We end this section by recalling the topological significance of the ring of stably numerical polynomials. This involves the determination of the cooperation algebra for complex K-theory, KU_*KU, discussed in [3, 11].

Theorem 7.3. *Let $u = \eta_L(t)$ and $v = \eta_R(t)$ be the images of the Bott generator $t \in KU_2$ under the left and right units $KU_* \longrightarrow KU_*KU$, and let $w = vu^{-1} \in KU_0KU$. Then the image of the (monomorphic) rationalisation map*

$$KU_0KU \longrightarrow KU_0KU\mathbb{Q} \cong KU_0KU \otimes \mathbb{Q}$$

is equal to the ring of stably numerical polynomials $\mathrm{A}^{\mathrm{S}}(w)$. More generally, if $R \subseteq \mathbb{Q}$ is any subring, then the image of the localization $KU_0KUR \cong KU_0KU \otimes R$ under the rationalisation map

$$KU_0KUR \longrightarrow KU_0KU \otimes \mathbb{Q}$$

is equal to the ring of stably R-numerical polynomials $\mathrm{A}^{\mathrm{S}}(w; R)$.

The natural Hopf algebroid structure on the pair (KU_*KU, KU_*) is then induced by the left and right units together with the maps

(coproduct)
$$u \longmapsto u \otimes 1;\ v \longmapsto 1 \otimes v;\ w \longmapsto w \otimes w;$$
(antipode)
$$u \longmapsto v;\ v \longmapsto u;\ w \longmapsto w^{-1},$$

where the coproduct is a ring homomorphism

$$KU_*KU \longrightarrow KU_*KU \underset{KU_*}{\otimes} KU_*KU$$

into the tensor product of bimodules obtain using the right-left KU_* module structures.

This result provides a model for our description of the cooperation algebra $E\ell\ell_*E\ell\ell$.

8. Katz's work on divided congruences amongst modular forms

In this section we will describe briefly results from N. Katz's paper [23], especially section 5. These will be applied to determine the rings of generalized modular forms.

Let $p > 3$ be a prime. We will work with the ring $\mathrm{S}(\mathbb{Z}_{(p)})_*$ of holomorphic modular forms defined over the ring of p-local integers $\mathbb{Z}_{(p)}$. For the remainder of this section we let $\mathrm{S}(\mathbb{Z}_{(p)})_\oplus$ denote the subring of $\mathbb{Z}_{(p)}[[q]]$ generated by the images of all of the individual gradings $\mathrm{S}(\mathbb{Z}_{(p)})_k$ under the homomorphism

$$\mathrm{S}(\mathbb{Z}_{(p)})_k \xrightarrow{\mathrm{eval}} \mathbb{Z}_{(p)}[[q]]; \quad F \longmapsto \widetilde{F}(q)$$

which assigns to each modular form its q-expansion. Clearly this is a polynomial subring $\mathbb{Z}_{(p)}[\widetilde{Q}, \widetilde{R}]$ of $\mathbb{Z}_{(p)}[[q]]$. However, it is not a direct summand as a $\mathbb{Z}_{(p)}$-module, as the congruence $1 - \widetilde{E}_{p-1} \equiv 0 \pmod p$ shows. For each $k \geqslant 1$, we will describe the kernel of the composition

$$\mathrm{eval}_{p^k} : \mathrm{S}(\mathbb{Z}_{(p)})_\oplus \xrightarrow{\mathrm{eval}} \mathbb{Z}_{(p)}[[q]] \xrightarrow{\mathrm{red}} \mathbb{Z}/p^k[[q]].$$

Definition 8.1. Define the numerical function h by

$$h(r) = \frac{p^r - 1}{p - 1} \quad \text{if } r \geqslant 1,$$

and $h(0) = 0$.

Theorem 8.2. *There is sequence of elements $R_0 = p, R_1, \ldots, R_k, \ldots$ in $\mathrm{S}(\mathbb{Z}_{(p)})_\oplus$ such that*

(1) *each R_k is a sum of the q-expansions of modular forms of weight at most $p^k - 1$;*
(2) *for each $k \geqslant 1$ there is a element $R'_k \in \mathbb{Z}_{(p)}[[q]]$ such that*

$$R_k = p^{h(k)} R'_k$$

in $\mathbb{Z}_{(p)}[[q]]$;
(3) *the evaluation modulo p^k map, eval_{p^k}, has as its the kernel the ideal $I_k \lhd \mathrm{S}(\mathbb{Z}_{(p)})_\oplus$ generated by the elements*

$$R_0^{r_0} R_1^{r_1} \cdots R_d^{r_d}$$

for which

$$r_0 + \sum_{1 \leqslant j \leqslant d} r_j h(j) \geqslant k.$$

In fact, in his theorem 5.5, Katz gives gives an explicit construction for the elements R'_k and R_k, and we will make use of this in Section 9. We define *the ring of (p-local) divided congruences* to be

$$\mathrm{DC}_p = \left\{ \Theta \in \mathbb{Q}[\widetilde{Q}, \widetilde{R}] : \Theta(q) \in \mathbb{Z}_{(p)}[[q]] \right\}.$$

Theorem 8.3. *The $\mathbb{Z}_{(p)}$-algebra DC_p is generated by the elements $\widetilde{Q}, \widetilde{R}$ and the R'_k $(k > 0)$. As a $\mathbb{Z}_{(p)}[\widetilde{Q}, \widetilde{R}]$-module, it is spanned by the elements*

$$R_0'^{r_0} R_1'^{r_1} \cdots R_d'^{r_d}$$

for which

$$r_0 + \sum_{1 \leqslant j \leqslant d} r_j h(j) \geqslant k.$$

There is an action of the p-local units $\mathbb{Z}_{(p)}^{\times}$ on the ungraded ring of modular forms $\mathbb{Q}[\widetilde{Q}, \widetilde{R}] \subseteq \mathbb{Q}[[q]]$, namely that given by

$$a \cdot \left(\sum_k \widetilde{F}_k \right) = \sum_k a^k \widetilde{F}_k,$$

where F_k has weight k. This action ultimately comes from the operation of including each lattice L into $(1/N)L$, for any natural number N, and is related to the elliptic cohomology Adams operations of [6].

Proposition 8.4. *The action of $\mathbb{Z}_{(p)}^{\times}$ on $\mathbb{Q}[\widetilde{Q}, \widetilde{R}]$ restricts to an action on the subring DC_p. Moreover, the eigenspaces of this action are the submodules of homogeneous weight modular forms.*

This is implicitly demonstrated by Katz in [23]. The second statement means that for $X \in \mathbb{Q}[\widetilde{Q}, \widetilde{R}]$,

$$\forall a \in \mathbb{Z}_{(p)}^{\times},\ a \cdot X = a^k X \iff X \text{ is the image of a weight } k \text{ modular form over } \mathbb{Q}.$$

We may view each element $\Theta \in \mathrm{DC}_p$ as defining a function

$$\mathbb{Z}_{(p)}^{\times} \longrightarrow \mathbb{Z}_{(p)}[[q]],$$

and thus we have

$$(a \cdot \Theta)(q) = \sum_{n \geqslant 0} c_n(a) q^n,$$

where the coefficient functions c_n are rational polynomial functions in a taking values in $\mathbb{Z}_{(p)}$, i.e., each c_n lies in the ring of semi-numerical polynomials $\mathrm{A}_0^{\mathrm{S}}(w; \mathbb{Z}_{(p)})$. One interpretation of this is in terms of the embedding $\mathrm{DC}_p \longrightarrow \mathbb{Q}[w][[q]]$ which sends Θ to $\sum_{n \geqslant 1} c_n(w) q^n$, and has image in the subring $\mathrm{A}_0^{\mathrm{S}}(w; \mathbb{Z}_{(p)})[[q]]$. Thus there is an embedding of rings

$$\mathrm{DC}_p \longrightarrow \mathrm{A}_0^{\mathrm{S}}(w; \mathbb{Z}_{(p)})[[q]]. \tag{8.1}$$

Notice that we can modify the definition of the ring of divided congruences to give a global version, namely

$$\mathrm{DC} = \left\{ \Theta \in \mathbb{Q}[\widetilde{Q}, \widetilde{R}] : \forall a \in \mathbb{Z} - \{0\},\ (a \cdot \Theta)(q) \in \mathbb{Z}[1/6a][[q]] \right\},$$

where we define the action of $a \in \mathbb{Z} - \{0\}$ similarly to the above action of $\mathbb{Z}_{(p)}^{\times}$. By viewing each Θ as a function $\mathbb{Z} - \{0\} \longrightarrow \mathbb{Q}[[q]]$, we see that there is an embedding of rings

$$\mathrm{DC} \longrightarrow \mathrm{A}_0^{\mathrm{S}}(w; \mathbb{Z}[1/6])[[q]]. \tag{8.2}$$

Of course, for either of rings DC_p and DC, we can get back from subrings of $\mathrm{A}^{\mathrm{S}}_0(w;\mathbb{Q})[[q]]$ to subrings of $\mathbb{Q}[[q]]$ by evaluating w at 1.

We end this section by remarking that although the element $\overline{\theta}_1(w) = (1-w^{p-1})/p$ can appear as the constant term of an element of DC_p, there is no element whose constant term is

$$w^{-1}\frac{(w-w^p)/p - ((w-w^p)/p)^p}{p}.$$

This is related to the fact that

$$\frac{(1-E_{p-1})/p - ((1-E_{p-1})/p)^p}{p}$$

is not a modular form modulo p in the sense of Serre, see [40].

We suspect that a direct proof of Katz's results (and equivalently of ours) should be possible making use of the ring of stably numerical polynomials, however at present this eludes us.

9. Calculation of the rings of generalized modular forms

In this section we determine the algebraic structure of the two rings of generalized modular forms

$${}^{\mathrm{Gen}}\mathrm{S}(\mathbb{Z}[1/6])_* \quad \text{and} \quad {}^{\mathrm{Gen}}\mathrm{M}(\mathbb{Z}[1/6])_*.$$

Our approach to this makes use of Katz's work which we have described in Section 8.

We are primarily interested in the (graded) ring ${}^{\mathrm{Gen}}\mathrm{M}(\mathbb{Z}[1/6])_*$, but it clearly suffices to consider the subring ${}^{\mathrm{Gen}}\mathrm{S}(\mathbb{Z}[1/6])_*$ consisting of holomorphic generalized modular forms. Now by Corollary 4.10, it suffices to determine the subring of $\mathbb{Q}[Q,R,Q',R']$ consisting of those homogeneous elements whose q-expansions lie in $\mathbb{Z}[1/6N,\zeta_N][[q,q']]$ whenever we evaluate on a pair of the form $L = \langle \tau, 1\rangle \subseteq L'$ with index N. Here $Q(L\subseteq L') = E_4(L)$, $R(L\subseteq L') = E_6(L)$ (a modular form in L alone), $Q'(L\subseteq L')) = E_4(L')$ and $R'(L\subseteq L')) = E_6(L')$ (a modular form in L' alone). Let us examine these conditions in more detail.

Now let $\Phi \in {}^{\mathrm{Gen}}\mathrm{S}(\mathbb{Z}[1/6])_n$ and $N \geqslant 1$. Let us evaluate Φ at a pair of lattices $L = \langle \tau, 1\rangle \subseteq L'$ with $[L;L'] = N$ and $\tau \in \mathfrak{H}$; notice that $L' \subseteq \langle \tau/N, 1/N\rangle$. Our data gives rise to an element of $\mathbb{Z}[1/6N,\zeta_N]((q^{1/N}))$. It is easily seen that

$$L' = \left\langle \frac{r\tau+t}{N}, \frac{s}{N}\right\rangle$$

for $0 \leqslant r,s,t \in \mathbb{Z}$ satisfying $rs = N$ and $0 \leqslant t < s$. Notice that given L, τ is unique to within an integer summand, and hence the element $\tau' = (r\tau+t)/N \in \mathfrak{H}$ is unique up to a summand of the form kr/N. Now suppose that we have the following expression for $\Phi \in \mathbb{Q}[Q,R,Q',R']_n$,

$$\Phi = \sum_{m,a,b} c_{m,a,b} F_{m,a} F'_{n-m,b}, \tag{9.1}$$

with $c_{m,a,b} \in \mathbb{Q}$, and $F_{m,a} \in \mathrm{M}(\mathbb{Z}[1/6])_m$, $F'_{n-m,b} \in \mathrm{S}(\mathbb{Z}[1/6])_{n-m}$ being taken from the standard basis of Section 1 evaluated on L and L'. Then we have

$$\Phi(L\subseteq L') = \sum_{m,a,b} c_{m,a,b} r^{n-m} \mathrm{eval}_q(F_{m,a}) \mathrm{eval}_{q'}(F'_{n-m,b}),$$

where

$$q' = e^{2\pi i\tau'}.$$

Thus our integrality condition on Φ amounts to the requirement that this series in q, q' has coefficients in $\mathbb{Z}[1/6N]$ for all $r|N$.

For a modular form $F\colon \mathcal{L} \longrightarrow \mathbb{C}$, let $\widetilde{F}\colon \mathfrak{H} \longrightarrow \mathbb{C}$ denote the q-series of the corresponding function on the upper half plane. Thus we have

$$\Phi(\langle \tau, 1\rangle \subseteq L') = \sum_{m,a,b} c_{m,a,b} r^{n-m} \widetilde{F}_{m,a}(q)\widetilde{F}_{n-m,b}(q'). \tag{9.2}$$

Notice that τ and q vary over infinite sets, and given τ, we may vary τ', and hence q', over infinite sets. Thus we can view $\Phi(L, L')$ as an element of $\mathbb{Z}[1/6N][[q, q']]$. The coefficients of monomials $q^i {q'}^j$ are rational polynomials $g_{i,j}(r)$ in r which also live in $\mathbb{Z}[1/6N]$ for all $r|N$. Since N (hence r) ranges over an infinite set, the polynomials $g_{i,j}(w) \in \mathbb{Q}[w]$ are uniquely determined by Φ; in fact they are in $\mathrm{A}_0^{\mathrm{S}}(w; \mathbb{Z}[1/6])$ (consider the case $r = N$). We have established the next theorem.

Theorem 9.1. *Evaluation at pairs $L \subseteq L'$ of index N and having the form*

$$L = \langle \tau, 1\rangle \subseteq L' = \left\langle \frac{r\tau + t}{N}, \frac{s}{N} \right\rangle \subseteq \langle \tau/N, 1/N\rangle \quad (0 \leqslant r, s, t,\ rs = N,\ 0 \leqslant t < s),$$

induces embeddings of (ungraded) rings

$$\begin{aligned} {}^{\mathrm{Gen}}\mathrm{S}(\mathbb{Z}[1/6])_* &\longrightarrow \mathrm{A}_0^{\mathrm{S}}(w; \mathbb{Z}[1/6])[[q, q']], \\ {}^{\mathrm{Gen}}\mathrm{M}(\mathbb{Z}[1/6])_* &\longrightarrow \mathrm{A}^{\mathrm{S}}(w; \mathbb{Z}[1/6])((q, q')), \end{aligned}$$

which in weight n yield embeddings

$$\begin{aligned} {}^{\mathrm{Gen}}\mathrm{S}(\mathbb{Z}[1/6])_n &\longrightarrow \mathrm{A}_0^{\mathrm{S}}(w; \mathbb{Z}[1/6])[[q, q']], \\ {}^{\mathrm{Gen}}\mathrm{M}(\mathbb{Z}[1/6])_n &\longrightarrow \mathrm{A}^{\mathrm{S}}(w; \mathbb{Z}[1/6])((q, q')). \end{aligned}$$

Setting the variable q equal to zero gives homomorphisms into the ring of divided congruences DC of Section 8. After localizing at a prime $p > 3$, we obtain $\mathbb{Z}_{(p)}$-module homomorphisms

$${}^{\mathrm{Gen}}\mathrm{S}(\mathbb{Z}_{(p)})_n \longrightarrow \mathrm{DC}_p \subseteq \mathrm{A}_0^{\mathrm{S}}(w; \mathbb{Z}_{(p)})[[q']], \tag{9.3}$$

$${}^{\mathrm{Gen}}\mathrm{M}(\mathbb{Z}_{(p)})_n \longrightarrow \mathrm{DC}_p[\widetilde{\Delta}^{-1}] \subseteq \mathrm{A}^{\mathrm{S}}(w; \mathbb{Z}_{(p)})((q')). \tag{9.4}$$

The rings generated by the images of all these maps are equal to the ring of divided congruences and its localization at powers of $\widetilde{\Delta}^{-1}$, as we shall see.

Now from Section 2, we see that there is a unique ring homomorphism

$$MU_*MU \longrightarrow {}^{\mathrm{Gen}}\mathrm{S}(\mathbb{Z}[1/6])_* \subseteq {}^{\mathrm{Gen}}\mathrm{M}(\mathbb{Z}[1/6])_*$$

extending the two homomorphisms

$$MU_* \longrightarrow \mathrm{S}(\mathbb{Z}[1/6])_* \underset{\eta_R}{\overset{\eta_L}{\rightrightarrows}} {}^{\mathrm{Gen}}\mathrm{S}(\mathbb{Z}[1/6])_* \subseteq {}^{\mathrm{Gen}}\mathrm{M}(\mathbb{Z}[1/6])_*$$

and classifying the universal isomorphism $H(T) \in {}^{\mathrm{Gen}}\mathrm{S}(\mathbb{Z}[1/6])_*[[T]]$ between the two Weierstrass formal group laws induced by the latter. Let

$$\log^{E\ell\ell} T = \sum_{n\geqslant 1} \frac{L_n}{n+1} T^{n+1},$$
$$\log^{E\ell\ell'} T = \sum_{n\geqslant 1} \frac{L'_n}{n+1} T^{n+1}$$

be the logarithms of these two formal group laws over ${}^{\mathrm{Gen}}\mathrm{S}(\mathbb{Z}[1/6])_*$. It is well-known that L_n and L'_n lie in ${}^{\mathrm{Gen}}\mathrm{S}(\mathbb{Z}[1/6])_n$.

Now there is a unique expression

$$B(T) = \sum_{k\geqslant 1}^{E\ell\ell'} H_k T^{k+1} \in {}^{\mathrm{Gen}}\mathrm{S}(\mathbb{Z}[1/6])_*[[T]], \tag{9.5}$$

with H_k having weight k. The H_k can be determined inductively using the equation

$$\log^{E\ell\ell'} H(T) = \log^{E\ell\ell} T,$$

which yields

$$L_{n-1} = \sum_{m|n} \frac{n}{m} L'_{m-1} H^m_{n/m-1}. \tag{9.6}$$

In particular, if p is a prime, we have

$$L_{p^r-1} = \sum_{0\leqslant s\leqslant r} p^{r-s} L'_{p^s-1} H^{p^s}_{p^{r-s}-1} \tag{9.7}$$

in ${}^{\mathrm{Gen}}\mathrm{S}(\mathbb{Z}[1/6])_* \subseteq {}^{\mathrm{Gen}}\mathrm{S}(\mathbb{Z}_{(p)})_*$.

This motivates us to define (for given prime $p > 3$)

$$A_r = L_{p^r-1},$$
$$A'_r = L'_{p^r-1},$$
$$D_r = H_{p^r-1}.$$

Thus we have in ${}^{\mathrm{Gen}}\mathrm{S}(\mathbb{Z}[1/6])_* \subseteq {}^{\mathrm{Gen}}\mathrm{S}(\mathbb{Q})_*$,

$$\begin{aligned} D_0 &= 1, \\ D_1 &= \frac{A_1 - A'_1}{p}, \\ D_2 &= \frac{A_2 - pA'_1 D_1^p - A'_2}{p^2}, \\ &\ \vdots \\ D_r &= \frac{A_r - p^{r-1}A'_1 D^p_{r-1} - p^{r-2} A'_2 D^{p^2}_{r-2} - \cdots - A'_r}{p^r}, \\ &\ \vdots \end{aligned} \tag{9.8}$$

The following is closely related to Katz [23], theorem 5.5, and is easily established by induction on r.

Proposition 9.2. *For $r \geqslant 1$,*

$$D'_r = p^{h(r)} D_r \in \mathbb{Z}[1/6][Q, R, Q', R'] \subseteq {}^{\mathrm{Gen}}\mathrm{S}(\mathbb{Z}[1/6])_*.$$

Notice also that if we expand $D_r(\langle \tau, 1\rangle \subseteq L')$ where $\langle \tau, 1\rangle \subseteq L'$ with index N as above, in the form of Equation 9.2, we obtain

$$\sum_{0 \leqslant n \leqslant p^r - 1} c_{p^r-1-n,0,0} w^n = \overline{\theta}_r(w). \tag{9.9}$$

and we also obtain a series in $\mathbb{Z}[1/6N][[q, q']]$ which on setting $q = 0$ yields an element of $\mathbb{Z}[1/6N][[q']]$. This maps each D_r to an element R'_r which is Katz's choice of generator as explained in Section 8 (with q replaced by q').

We will prove the following theorem.

Theorem 9.3. *For each prime $p > 3$, the ring ${}^{\mathrm{Gen}}\mathrm{S}(\mathbb{Z}_{(p)})_*$ is generated as an algebra over $\mathrm{S}(\mathbb{Z}_{(p)})_*$ by the elements D_r, $r \geqslant 1$, together with Q' and R'. Similarly, as an algebra over $\mathrm{M}(\mathbb{Z}_{(p)})_*$, ${}^{\mathrm{Gen}}\mathrm{M}(\mathbb{Z}_{(p)})_*$ is generated by the elements D_r, $r \geqslant 1$ together with Q', R' and ${\Delta'}^{-1}$, i.e.,*

$${}^{\mathrm{Gen}}\mathrm{M}(\mathbb{Z}_{(p)})_* = {}^{\mathrm{Gen}}\mathrm{S}(\mathbb{Z}_{(p)})_*[Q', R', \Delta^{-1}, {\Delta'}^{-1}].$$

Proof. We will prove Theorem 9.3 for ${}^{\mathrm{Gen}}\mathrm{S}(\mathbb{Z}_{(p)})_*$ by induction upon the weight $\mathrm{wt}\,\Phi$ of an element. Clearly the weight 0 case is true, so assume that whenever $\mathrm{wt}\,\Phi < n$, Φ is expressible as a polynomial in the generators indicated.

Now assume that $\mathrm{wt}\,\Phi = n$. Then Φ can be expressed in the form indicated in Equation 9.1 and 9.2:

$$\Phi = \sum_{m,a,b} c_{m,a,b} F_{m,a} F'_{n-m,b}.$$

On taking q-expansions in the manner of Theorem 9.1, we have

$$\widetilde{\Phi} = \sum_{m,a,b} c_{m,a,b} w^{n-m} \widetilde{F}_{m,a} \widetilde{F}'_{n-m,b}.$$

By setting $q = 0$, we obtain a q'-expansion

$$\widetilde{\Phi}(0, q') = \sum_{m,a,b} c_{m,a,b} \widetilde{F}'_{n-m,b}$$

lying in $\mathrm{DC}_p \subseteq \mathrm{A}^{\mathrm{S}}_0(w; \mathbb{Z}_{(p)})[[q']]$. Now by Theorem 8.3, this can be expressed as a polynomial in the elements $\widetilde{Q}$, $\widetilde{R}$ and $\widetilde{R}'_k$ ($k \geqslant 1$) (evaluated at q' rather than q). Now construct a (non-homogeneous) element of ${}^{\mathrm{Gen}}\mathrm{S}(\mathbb{Z}_{(p)})_*$ as follows.

First replace each occurrence of R'_k in $\widetilde{\Phi}(0, q')$ by the element $D_k \in {}^{\mathrm{Gen}}\mathrm{S}(\mathbb{Z}_{(p)})_*$ defined in Equation 9.8. This will be a sum of homogeneous terms Θ_d of weights d in the range $0 \leqslant d \leqslant n$. Now multiply Θ_d by the basis element $F_{n-d,0}$ to get an element $F_{n-d,0}\Theta_d$ which has weight n. Let

$$\Phi_0 = \sum_{0 \leqslant d \leqslant n} F_{n-d,0}\Theta_d.$$

Notice that we have

$$\widetilde{\Phi}(0, q') - \widetilde{\Phi}_0(0, q') = 0,$$

and hence we have

$$\Phi = \Phi_0 + \Delta\Phi'$$

in the ring ${}^{\mathrm{Gen}}\mathrm{S}(\mathbb{Z}_{(p)})_*$. Hence, we can appeal to the inductive assumption to express $\Phi' \in {}^{\mathrm{Gen}}\mathrm{S}(\mathbb{Z}_{(p)})_{n-12}$ in the required form. Thus, Φ is also of the required form and we have completed the inductive step.

This completes the proof of Theorem 9.3. □

As an immediate consequence we obtain our desired global result.

Theorem 9.4. *As an algebra over* $\mathrm{S}(\mathbb{Z}[1/6])_*$*, the ring* ${}^{\mathrm{Gen}}\mathrm{S}(\mathbb{Z}[1/6])_*$ *is generated by the elements* H_n*,* $n \geqslant 1$*; similarly, as an algebra over* $\mathrm{M}(\mathbb{Z}[1/6])_*$*,* ${}^{\mathrm{Gen}}\mathrm{M}(\mathbb{Z}[1/6])_*$ *is generated by the elements* H_n*,* $n \geqslant 1$ *together with* ${\Delta'}^{-1}$*, i.e.,*

$${}^{\mathrm{Gen}}\mathrm{M}(\mathbb{Z}[1/6])_* = {}^{\mathrm{Gen}}\mathrm{S}(\mathbb{Z}[1/6])_*[\Delta^{-1}, {\Delta'}^{-1}].$$

Hence there is an isomorphism of algebras over $E\ell\ell_* \cong \mathrm{M}(\mathbb{Z}[1/6])_*$*,*

$$E\ell\ell_* E\ell\ell \cong {}^{\mathrm{Gen}}\mathrm{M}(\mathbb{Z}[1/6])_*.$$

The proof of Theorem 9.3 actually shows the following, which should be compared with the result of Katz, Theorem 8.2. Recall the element $D'_r = p^{h(r)} D_r$ of Proposition 9.2.

Theorem 9.5. *An element* $\Phi \in E\ell\ell_* E\ell\ell_{(p)}$ *has* q, q'*-expansion in* $p^k\mathbb{Z}((q, q'))$ *if and only if* Φ *is in the ideal generated by*

$$p^{r_0} {D'_1}^{r_1} \cdots {D'_d}^{r_d}$$

for which

$$r_0 + \sum_{1 \leqslant j \leqslant d} r_j h(j) \geqslant k.$$

10. The cooperation algebra as a Hopf algebroid

In this section we complete our description of the cooperation algebra by describing the Hopf algebroid structure in terms of generalized modular forms. The existence of the Hopf algebroid structure over $\mathbb{Z}[1/6]$ follows the topological result for $E\ell\ell_* E\ell\ell$. An element $\Phi \in E\ell\ell_{2n} E\ell\ell$ is equivalent to a generalized modular form

$$(F_\bullet \colon \mathcal{L}^\bullet \longrightarrow \mathbb{C}) \in {}^{\mathrm{Gen}}\mathrm{M}(\mathbb{Z}[1/6])_n$$

with certain properties. At the end of Section 4, a partial monoid structure

$$\mu \colon \mathcal{L}^\bullet \times_{\mathcal{L}} \mathcal{L}^\bullet \longrightarrow \mathcal{L}^\bullet$$

was described. This induces a coproduct

$$F_\bullet \longmapsto F_\bullet \circ \mu$$

which is actually a ring homomorphism

$$\psi\colon {}^{\mathrm{Gen}}\mathrm{M}(\mathbb{Z}[1/6])_* \longrightarrow {}^{\mathrm{Gen}}\mathrm{M}(\mathbb{Z}[1/6])_* \underset{\mathrm{M}(\mathbb{Z}[1/6])_*}{\otimes} {}^{\mathrm{Gen}}\mathrm{M}(\mathbb{Z}[1/6])_*,$$

where the tensor product involves the right and left $\mathrm{M}(\mathbb{Z}[1/6])_*$-module structures. This is derived ultimately from the composition of lattice inclusions $L \subseteq L'$ and $L' \subseteq L''$ to give $L \subseteq L''$; then

$$F_\bullet\mu(L' \subseteq L'', L \subseteq L') = F_\bullet(L \subseteq L'').$$

There is also an antipode map, which arises as follows. Let $L \subseteq L'$ with index N. Then there is a *dual isogeny* $L' \subseteq (1/N)L$, also of index N, and this can be scaled to give the inclusion $N \cdot L' \subseteq L$. We can evaluate a generalized modular form $F_\bullet$ of weight n on this inclusion to obtain a function of the form

$$(L \subseteq L') \longmapsto F_\bullet(N \cdot L' \subseteq L).$$

Writing

$$F_\bullet = \sum_r F_r F'_{n-r},$$

where

$$F_\bullet(L, L') = \sum_r F_r(L) F'_{n-r}(L'),$$

this is the same as the function

$$(L \subseteq L') \longmapsto \sum_r N^{-r} F_r(L') F'_{n-r}(L).$$

We then define action of the antipode χ on $F_\bullet$ by

$$\chi F_\bullet(L, L') = F_\bullet(N \cdot L' \subseteq L).$$

Thus we may loosely say that the antipode is induced by inverting each inclusion $L \subseteq L'$ and evaluating on its inverse.

It would be interesting to give a purely algebraic proof that the coproduct ψ actually lands in the tensor product over $\mathrm{M}(\mathbb{Z}[1/6])_*$, since although it is clear that the rationalisation behaves correctly, the arithmetic conditions appear subtle. Of course, we can appeal to the topological fact that $E\ell\ell_* E\ell\ell$ is a Hopf algebroid to obtain this. A similar problem occurs with the ring of stably numerical polynomials $\mathrm{A}^{\mathrm{S}}(w;\mathbb{Z})$, which is a Hopf algebra over $\mathbb{Z}$, but the easiest proof of this uses the topological gadget KU_*KU.

We can interpret this Hopf algebroid as a Hopf algebroid of functions on the category $\mathbf{Iso}_{\mathbb{C}}$, see Proposition 5.1. More generally, we have the following (see Theorem 5.2 for the categorical localization result).

Theorem 10.1. *Let R be a subring of $\mathbb{Q}$ containing $1/6$. Then we may identify ${}^{\mathrm{Gen}}\mathrm{M}(R)_*$ with the ring of generalized modular forms on $\mathcal{L}^\bullet$ which extend to functions on $\mathbf{SIso}_{\mathbb{C}}^{R^\times_+} \supseteq \mathbf{SIso}_{\mathbb{C}}$ which have q-expansions defined over R. Morover, composition and inversion in $\mathbf{SIso}_{\mathbb{C}}^{R^\times_+}$ give rise to the natural Hopf algebroid structure on ${}^{\mathrm{Gen}}\mathrm{M}(R)_*$.*

It is interesting to compare this with the corresponding situation for stably numerical polynomials; there we have

Proposition 10.2. *For any subring* $K \subseteq \mathbb{Q}$,

$$\mathrm{A}^{\mathrm{S}}(w, K) \subseteq \{f(w) \in \mathbb{Q}[w, w^{-1}] : \forall u \in K^{\times},\ f(u) \in K\}.$$

In particular, for any prime p,

$$\mathrm{A}^{\mathrm{S}}(w, \mathbb{Z}_{(p)}) = \{f(w) \in \mathbb{Q}[w, w^{-1}] : \forall u \in \mathbb{Z}_{(p)}^{\times},\ f(u) \in \mathbb{Z}_{(p)}\}.$$

Of course, Theorem 10.1 gives a similar interpretation for the Hopf algebroid $E\ell\ell_* E\ell\ell R$.

11. Operations dual to cooperations

In this section we will briefly describe how our knowledge of $E\ell\ell_* E\ell\ell$ gives information about stable operations in elliptic cohomology. For any subring $R \subseteq \mathbb{Q}$ containing $1/6$, the Universal Coefficient spectral sequence described in Equation 1.3 applied to the case where $M = E\ell\ell R$ and $X = E\ell\ell$ gives

$$\mathrm{E}_2^{*,*}(E\ell\ell) = \mathrm{Ext}^{*,*}_{E\ell\ell_*}(E\ell\ell_*(E\ell\ell), E\ell\ell R_*) \implies E\ell\ell R^*(E\ell\ell). \tag{11.1}$$

As $E\ell\ell_*$ is a ring of dimension 2, we know that

$$\mathrm{Ext}^{k,*}_{E\ell\ell_*} = 0 \quad \text{if } k > 2.$$

Hopkins and Hunton's work as described in 1.2 together with the Milnor exact sequence yields

$$E\ell\ell R^*(E\ell\ell) \cong \lim_{\alpha} E\ell\ell R^*(E_\alpha),$$

where the E_α form a cofinal collection of finite CW subspectra of $E\ell\ell$. Thus stable operations $E\ell\ell^*(\ ) \longrightarrow E\ell\ell R^*(\ )$ determine unique morphisms of spectra $E\ell\ell \longrightarrow E\ell\ell R$ from their values on finite CW spectra.

Now to construct stable operations it suffices to write down natural transformations $E\ell\ell^*(\ ) \longrightarrow E\ell\ell R^*(\ )$ defined on the category of finite CW spectra; the most accessible type of these arise as follows. We use the coaction map

$$\psi\colon E\ell\ell_* E\ell\ell \longrightarrow E\ell\ell_* E\ell\ell \underset{E\ell\ell_*}{\otimes} E\ell\ell_* E\ell\ell$$

which is left $E\ell\ell_*$ linear. Given any left $E\ell\ell_*$-linear mapping

$$\Theta\colon E\ell\ell_* E\ell\ell \longrightarrow E\ell\ell R_*$$

we obtain an operation as the composite

$$\begin{aligned} \overline{\Theta}\colon E\ell\ell^*(\ ) \cong (S^0 \wedge E\ell\ell)^*(\ ) &\longrightarrow (E\ell\ell \wedge E\ell\ell)^*(\ ) \\ &\xrightarrow{\cong} E\ell\ell_* E\ell\ell \underset{E\ell\ell_*}{\otimes} E\ell\ell^*(\ ) \\ &\xrightarrow{\Theta} E\ell\ell R_* \underset{E\ell\ell_*}{\otimes} E\ell\ell^*(\ ) \\ &\longrightarrow E\ell\ell R^*(\ ). \end{aligned} \tag{11.2}$$

This is the construction underlying the Adams and Hecke operations described in Equations 6.4 and 6.5, based on [6]. We will return to this in Part II, where we will view $E\ell\ell_*E\ell\ell$ as a kind of dual object to a Hecke algebra. Once again, this closely follows the situation for KU_*KU, which can be thought of as a sort of dual to the monoid ring $\mathbb{Z}[\mathbb{Z}-\{0\}]$.

This approach to stable operations in elliptic cohomology becomes more manageable if we reduce modulo an invariant ideal in the coefficient ring $E\ell\ell_*$. Such ideals were considered in [7]. The most interesting examples are of the form

$$\begin{aligned} J_{p,1} &= (p) \quad \text{and} \quad J_{p,1}^r, \quad r \geqslant 1 \\ J_{p,2} &= (p, E_{p-1}) \quad \text{and} \quad J_{p,2}^s, \quad s \geqslant 1, \end{aligned} \tag{11.3}$$

where $p > 3$ is a prime. Actually the second example consists of ideals in the p-localization $(E\ell\ell_*)_{(p)}$ since $E_{(p-1)}$ may only exist p-locally. We can form completions with respect to such ideals, and the reductions $E\ell\ell_*E\ell\ell/I$ and their completions $\widehat{E\ell\ell_*E\ell\ell_I}$ have interpretations as rings of continuous functions on completions of Hecke algebras and their underlying monoids. Again, this is parallel to known constructions for reduction modulo p^k and p-adic completion of KU_*KU which gives spaces of continuous functions on the group of p-adic units $\mathbb{Z}_{(p)}^\times$ and its pro-group ring $\mathbb{Z}_p[\mathbb{Z}_{(p)}^\times]$.

We end this section with some remarks on the Adams spectral sequence in elliptic homology. As usual for good homology theories, there is a spectral sequence of the form

$$\begin{cases} \mathrm{E}_2^{*,*}(X) & \Longrightarrow \pi_*(\mathrm{L}_{E\ell\ell}X); \\ \mathrm{E}_2^{*,*}(X) & = \mathrm{Ext}_{E\ell\ell_*E\ell\ell}^{*,*}(E\ell\ell_*, E\ell\ell_*(X)), \end{cases} \tag{11.4}$$

where the Ext functor is defined on the category of comodules over $E\ell\ell_*E\ell\ell$. Using the above families of ideals there are various 'chromatic' approaches to calculating this E_2-term and these may be interesting to pursue. For example, in [14], Clarke and Johnson have made some observations on the K-theoretic part of the 1-line $\mathrm{E}_2^{1,*}$, using Serre's theory of p-adic modular forms. This p-adic theory is discussed in [41] and its elliptic cohomology version in [5]. For the supersingular theory of modular forms, see [34], and also [7] for the topological version.

12. The operator of Halphen–Fricke–Ramanujan–Swinnerton-Dyer–Serre

The operator of the title has an interesting history; it plays a central rôle in the algebraic theory of the ring of modular forms. For our present purposes, it is an operator ∂ on the ring of modular forms which raises weight by 2, is a derivation and annihilates the discriminant Δ. For an early reference see [16], and for more recent descriptions see [24, 40, 22]. The congruence conditions in Section 9 ultimately rely upon arguments making use of ∂.

We have the following formulæ for the action of ∂:

$$\partial(Q) = R, \tag{12.1}$$

$$\partial(R) = \frac{3}{2}Q^2, \tag{12.2}$$

$$\partial(\Delta) = 0, \tag{12.3}$$

$$\partial(AB) = \partial(A)B + A\partial(B) \quad \text{if } A, B \in E\ell\ell_*. \tag{12.4}$$

Notice that multiplication by Δ (the periodicity operator in elliptic cohomology) commutes with ∂. Thus the following conjecture may seem reasonable.

Conjecture 12.1. The derivation ∂ extends to a stable operation on elliptic cohomology $E\ell\ell^*(\ )$.

The fact that ∂ plays a major rôle in the algebraic theory of the rings $e\ell\ell_*$ and $E\ell\ell_*$ also make this conjecture interesting. However, Conjecture 12.1 is actually false.

Theorem 12.2. *Let $p > 3$ be a prime. Then there is no* stable *operation $E\ell\ell^*(\ ) \longrightarrow E\ell\ell^*_{(p)}(\ )$ raising degree by 4 and extending ∂ on the coefficient ring $E\ell\ell_{-*}$. Hence there is no stable operation $E\ell\ell^*(\ ) \longrightarrow E\ell\ell^*(\ )$ raising degree by 4 and extending ∂ on $E\ell\ell_{-*}$.*

Proof. Suppose that such a stable operation $\overline{\partial}$ exists; then there is a corresponding morphism of spectra $\partial\colon E\ell\ell \longrightarrow \Sigma^{-4}E\ell\ell_{(p)}$ inducing $\overline{\partial}$ as a natural transformation of representable functors $E\ell\ell^*(\ ) \longrightarrow E\ell\ell^*_{(p)}(\ )$. We can extend ∂ to a morphism of $E\ell\ell$ module spectra

$$\partial^\dagger : E\ell\ell \wedge E\ell\ell \xrightarrow{1\wedge\partial} E\ell\ell \wedge \Sigma^{-4}E\ell\ell_{(p)} \xrightarrow{\mu_{E\ell\ell}} E\ell\ell_{(p)}$$

where $\mu_{E\ell\ell}\colon E\ell\ell \wedge E\ell\ell \longrightarrow E\ell\ell$ is the product map and its localization. Hence, we obtain a homomorphism of $E\ell\ell_*$ modules

$$\partial^\dagger_*\colon E\ell\ell_*(E\ell\ell) \longrightarrow (E\ell\ell_*)_{(p)}.$$

Notice that we also have a commutative diagram

$$\begin{array}{ccc} S^0 \wedge E\ell\ell & \xrightarrow{\ \simeq\ } & E\ell\ell \\ {\scriptstyle \eta\wedge 1}\big\downarrow & & {\scriptstyle \partial}\big\downarrow \\ E\ell\ell \wedge E\ell\ell & \xrightarrow{\ \partial^\dagger\ } & E\ell\ell_{(p)} \end{array}$$

where $\eta\colon S^0 \longrightarrow E\ell\ell$ is the unit for the ring spectrum $E\ell\ell$. But this means that the composite

$$E\ell\ell_* \xrightarrow{\eta_R} E\ell\ell_*(E\ell\ell) \xrightarrow{\partial^\dagger_*} (E\ell\ell_*)_{(p)}$$

agrees with ∂.

Now in the ring $E\ell\ell_*(E\ell\ell)_{(p)}$ we have an element of the form

$$\left(\frac{E_{p-1} - E'_{p-1}}{p}\right) \in E\ell\ell_{2(p-1)}(E\ell\ell)_{(p)} \subseteq E\ell\ell_{2(p-1)}(E\ell\ell) \otimes \mathbb{Q},$$

where $E'_{p-1} = \eta_R(E_{p-1})$. This follows from the well known fact that working modulo p in $(E\ell\ell_*)_{(p)}$, E_{p-1} agrees with the image of Hazewinkel generator $v_1 \in (MU_{2(p-1)})_{(p)}$ under the elliptic genus $(MU_*)_{(p)} \longrightarrow (E\ell\ell_*)_{(p)}$ (see [29] for example). But now applying the homomorphism $\partial^\dagger{}_*$ and we see that

$$\partial^\dagger_*\left(\frac{E_{p-1} - E'_{p-1}}{p}\right) = \frac{\partial(E_{p-1})}{p} \in (E\ell\ell_*)_{(p)} \subseteq E\ell\ell_* \otimes \mathbb{Q},$$

since

$$\partial^\dagger_*(E_{p-1}) = E_{p-1}\partial^\dagger_*(1) = 0.$$

However, from [41] we have

$$\begin{aligned}\partial(E_{p-1}) &\equiv \frac{1}{12}E_{p+1} \bmod p\\ &\not\equiv 0 \bmod p.\end{aligned}$$

Hence, this is an element of $E\ell\ell_* \otimes \mathbb{Q}$ which is not in $(E\ell\ell_*)_{(p)}$. $\square$

However, there is still the possibility of *unstable* extensions and we make the modified conjecture:

Conjecture 12.3. There are extensions of ∂ to unstable operations in elliptic cohomology $E\ell\ell^*(\ )$.

What is really meant here is that for a given $n \in \mathbb{Z}$, there might be a map

$$\Omega^{\infty-n}E\ell\ell \to \Omega^{\infty-n+4}E\ell\ell$$

inducing the operator ∂ in homotopy; however, such a map need not deloop.

An alternative approach is to try to construct a suitable stable operation locally at each prime. An obvious candidate would be an extension of the derivation ∂_p which raises weight by $p+1$ and is given by

$$\partial_p(F) = E_{p-1}\partial(F) - \frac{\mathrm{wt}(F)}{p-1}\partial(E_{p-1})F,$$

which has the property that $\partial_p(E_{p-1}) = 0$ and avoids the difficulties encountered with ∂. In fact, on q-expansions taken modulo p, ∂_p agrees with the action of $q\mathrm{d}/\mathrm{d}q$; it is thus the same as the operation θ studied by Serre and Swinnerton-Dyer on modular forms modulo p. However, it is still not clear if this extends to an operation taking values in elliptic cohomology modulo p; it also fails to commute with multiplication by Δ.

Finally, we note that in [18, 19], Gross and Hopkins have explored deformation theory for Lubin–Tate formal group laws; in particular they consider certain Gauss–Manin connections. Now it is known from [22, 24] that ∂ is also a Gauss–Manin connection, so there may well be some relationship between their work and the above discussion. We hope to return to these matters in future work.

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